LECTURE 24+1

1. Invariance via Commutativity
2. Eigenbasis of commuting transformations
3. The "all or nothing" theorem

III. COMMUTING PAIRS OF TRANSFORMATIONS.

1. Consider the following two linear transformation

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} : E^3 \to E^3$$

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} : E^3 \to E^3$$

K express a rotation around the x-axis by an angle θ.

L expresses a projection onto the y-z-plane combined with a rotation by α in that y-z-plane.

2. A simple computation shows that

$$LK = KL \left(= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos(\theta+\alpha) & -\sin(\theta+\alpha) \\ 0 & \sin(\theta+\alpha) & \cos(\theta+\alpha) \end{bmatrix} \right)$$

This pair of linear transformations and others like it highlight an important

(ii) if one of them has an eigenvector basis for U, then the other has the same eigenvectors.

property which is identified by the following

Definition 23.2 (Commutativity and the commutator)

Two linear transformations

$$K, L : U \to U$$

are said to commute if $KL = LK$.

The matrix $KL - LK \equiv [K, L]$ is called the commutator of K and L. Thus K and L commute if their commutor vanishes, i.e equals the zero transformation.

Comment

The importance of the commutativity of a pair of transformation is that

(i) the null space and the range space of one are invariant under the other.

Commutativity of a pair of linear transformation is the algebraic signal for the invariance of the subspaces of one transformation under the action of the other.

<u>Theorem</u> 2.4.1 (Invariance via Commutativity)

If $[K, L] \equiv KL - LK = 0$ then

I) $\mathcal{N}(K)$ is invariant under L

II) $\mathcal{R}(K)$ is invariant under L

Proof of I)

Let $x \in \mathcal{N}(K)$. We must show that $Lx \in \mathcal{N}(K)$, so that

$$L(\mathcal{N}(K)) \subset \mathcal{N}(K).$$

We have

$$Kx = 0;$$

Mply by L and obtain

$$0 = LKx = KLx; \therefore Lx \in \mathcal{N}(K)$$

$\mathcal{N}(K)$ is invariant under L indeed.

~~Proof of II~~

Let $x \in \mathcal{R}(K)$. We must show that $Lx \in \mathcal{R}(K)$, so that

$$L(\mathcal{R}(K)) \subset \mathcal{R}(K).$$

We have $x \in \mathcal{R}(K) \Rightarrow x$ has a preimage, namely $\exists$ a y s.t. $Ky = x$. $\} \in \mathcal{R}(K)$

Mply by L and obtain

$Lx \overset{?}{\in} \mathcal{R}(K)$

i.e $L(\mathcal{R}(K)) \overset{?}{\subseteq} \mathcal{R}(K)$

$$Lx = LKy = KLy$$

Thus $Lx \in \mathcal{R}(K)$ because Lx has a preimage, namely Ly s.t. $KLy = Lx$

In other words, $L(\mathcal{R}(K)) \subset \mathcal{R}(K)$;

$\mathcal{R}(K)$ is invariant under L indeed.

This theorem has an important consequence which is given by the following.

Corollary. (Common eigenvectors)

Let $[K, L] = 0$

Let L have an eigenvalue λ of finite multiplicity:

$$\dim \mathcal{N}(L - \lambda I) = k \geq 1$$

Then both transformations have a common eigen vector $\vec{x}$; in other words, $\exists \vec{x}$ such that

$$(L - \lambda I)\vec{x} = 0 \text{ and } K\vec{x} = \kappa \vec{x} \text{ for some } \kappa.$$

Proof. (in three steps)

Step I

$$Ly = \lambda y \Leftrightarrow (L - \lambda I)y = 0 \Leftrightarrow y \in \mathcal{N}(L - \lambda I)$$

Step II

According to Theorem 24.1,

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$\begin{pmatrix} \exists\, x \in \mathcal{M} \\ \text{s.t.} \\ Kx = \kappa x \end{pmatrix}$ for some κ $\Leftarrow \begin{cases} \mathcal{N}(L - \lambda I) \equiv \mathcal{M} \text{ is}_{\text{(a subspace)}} \text{ invariant under } K. \\ \dim \mathcal{M} = k \geq 1 \end{cases}$

Step III

(i) This invariance (under K) together with

(ii) $\dim \mathcal{M} = k \geq 1$

form the pair of hypotheses which according to Theorem 23.1 on Page 23.4 imply that $\exists\ x \in \mathcal{M} \equiv \mathcal{N}(L - \lambda I)$,

i.e $\vec{x}$ satisfies $\boxed{(L - \lambda I)\vec{x} = 0}$

such that

$$\boxed{K\vec{x} = \kappa\vec{x} \quad \text{for some } \kappa} \quad \text{Q.E.D.}$$

Concluding Remark (A Corollary)

Suppose

a) $K, L : U \to U$ $\quad \dim U = n$

are linear transformations with

L having n "non-degenerate" eigenvalues:

$\boxed{\lambda_i \text{ are all diff't.}}$

$$L\vec{x}_i = \lambda_i \vec{x}_i, \quad i = 1, \ldots, n; \quad \lambda_1 \neq \lambda_2$$

Thus each $\mathcal{M}_i = sp(\{x_i\})$, $i=1,\ldots,n$ is invar't. under L

b) $[K, L] = 0$ $\quad \dim(\mathcal{M}_i) = 1$

$\dim \mathcal{N}(L - \lambda_i) = 1$

Conclusion:

K has the same n eigenvectors

$$K\vec{x}_i = k_i \vec{x}_i, \quad i = 1, \ldots, n$$

In other words:

> If L has only non-degenererate eigenvalues, then it has an eigenvector basis basis and K has the same eigenvector basis for U
>
> whenever $[K, L] = 0$

Suppose we have two linear transformations L_1 and L_2 which have no common invariant subspaces except $\{0\}$ and U. Such transformation don't even have a single eigenvector in common.

A transformation K which commutes with both of such transformations has a surprising property. It is used, among others, repeatedly in the study of physical systems that exhibit invariance under symmetry transformations such as translations or rotations. This property is stated in the following

Theorem 24.2 ("All or nothing")

(i) Let L_1 and L_2 be linear transformations $U \rightarrow U$ whose *only* common invariant subspaces are $\{0\}$ and U = the whole vector space

(ii) Consider a K which commutes with L_1 and L_2: $[L_1, K] = 0$ and $[L_2, K] = 0$

Conclusion

Either *K maps every element into zero*
or
K has a unique inverse

Comment:

The theorem is known as "Schur's Lemma".

It plays a major role in theoretical mathematics (e.g. representation theory) and in theoretical physics (e.g. quantum and elementary particle theory).

Proof:

Thm on p 24.1

(a) $[L_1,K]=0 \Rightarrow \mathcal{N}(K)$ and $\mathcal{N}(R(K))$ are invariant under L_1

$[L_2,K]=0 \Rightarrow \mathcal{N}(K)$ and $\mathcal{N}(R(K))$ are invariant under L_2

invt under both L_1 and L_2

By hypothesis, the only subspaces invt under both L_1 & L_2 are $\{0\}$ and U

$$\mathcal{N}(K)=\{0\} \text{ or } \mathcal{N}(K)=U \quad \text{(the whole vector space)}$$

are the only subspace invt

Thus either

K is one-to-one or K maps every vector into zero

Note: No conclusion about the possibility of K being onto.

(b)

Similarly $\left\{\begin{matrix}[L_1,K]=0\\ [L_2,K]=0\end{matrix}\right\}$ imply

$$R(K)=\{0\} \text{ or } R(K)=U \quad \text{(the whole vector space)}$$

Thus, using $\dim \mathcal{N}(K)+\dim R(K)=n$, one has

"K maps every vector into zero" or "K is onto"

(c) (a) + (b) $\Rightarrow$ either K has an inverse

or K is the zero transform'n.