LECTURE 25

Inner Product Spaces

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Inner Product Spaces

I. Motivation

The feature common to all the vector space concepts introduced so far is that there was no commensurability along different dimensions. Different basis-induced coordinate lines would measure entirely different properties. Thus, the basis vector along the first coordinate line could measure the number of miles, the basis vector along the second the number of meters, the one along the third the number of lightyears, etc.

All this was before one introduced the concept of a metric (in LECTURE). There was no need for concepts such as the lengths of vectors or the angles between them.

The concept of a metric is an entirely distinct ("geometric") structure. It is the Pythagorean theorem $a^2+b^2=c^2$, generalized from Euclidean to those vector spaces whose coordinate lines are observed to be related by virtue of their comensurability. However, instead of restricting this generalization to real vector spaces, (Section 4.6 in Chapter 4 of J., R., & Arnold's 3rd Edition) we shall now do so to include also complex vector spaces. This is achieved by means of the following

II. Definition (Inner-Product Spaces)

An inner product on a complex vector space V is a bilinear, positive function such that for all vectors $u, v, w \in V$

a) $(u+v, w) = (u, w) + (v, w)$

b) $(u, cv) = c(u, v)$ $\forall$ complex number c

[Nota bene: the above is the physics/engineering convention. Some mathematics texts use the convention

$$(cu, v) = c(u, v)$$

instead.]

c) $(u, v) = \overline{(v, u)}$; (the bar denotes complex conjugation)

d) $(u, u) > 0$ whenever $u \neq 0$

Three comments:

(i) Property c) is necessary; otherwise one would have a contradiction with d):

$$(u,u) > 0 \Rightarrow (iu, iu) = i(iu, u) = i(u, iu) = -(u,u) < 0$$

(ii) $(cu, v) = \overline{(v, cu)} = \bar{c}\,\overline{(v, u)} = \bar{c}\,(u, v)$

Consequently,

b') $\boxed{(cu, v) = \bar{c}(u, v)}$

One says that (u, v) is anti-linear in the first factor, but is linear in the second factor.

(iii) Q: What is difference between a metric and an inner product?

A: A metric is more general in that

1) $g(u,u) = 0$ $\quad u \neq 0$

and

2) $g(u,u) < 0$

are allowed. However, a metric is also more special in that it presumes

1) V is a real vector-space, and

2) $\dim V =$ finite.

III. Examples of inner products.

Ex. 1

For $V=R^n$ let $x=[x_1,\dots,x_n]^T$

$y=[y_1,\dots,y_n]^T$.

Then the standard inner product is

$$(x,y) = x_1y_1+\dots+x_ny_n=[x_1,\dots,x_n]\begin{bmatrix}y_1\\ \vdots\\ y_n\end{bmatrix}\equiv x^Ty$$

Ex. 2 a)

Let $V=R^2$

Then $(x,y)=x_1y_1-x_2y_1-x_1y_2+4x_2y_2$

is a non-standard inner product

We do have positive-definiteness:

$$(x,x)=x_1^2-2x_1x_2+4x_2^2=(x_1-x_2)^2+3x_2^2$$

Ex. 2b

Let $G=\begin{bmatrix}1&-1\\-1&4\end{bmatrix}$; $V=R^2$;

then

$$(x,y)\equiv x^TGy=[x_1,x_2]\begin{bmatrix}1&-1\\-1&4\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}$$

$$=x_1y_1-x_2y_1-x_1y_2+4x_2y_2$$

is an inner product.

Ex. 4 (Different inner products on the same V)

Let $V=P_2$; $p(x), q(x)\in P_2$

$$(p,q)_1=p(0)q(0)+p(1)q(1)+p(2)q(2)$$

and

$$(p,q)_2=\int_0^1 p(t)\,q(t)\,dt$$

are two different inner products

Ex. 5

Let $V=$ set of continuous complex functions on $[a,b]$

$$(f,g)=\int_a^b \bar{f}(t)\,g(t)\,dt$$

IV. Orthogonality and Normalization

are geometrical concepts which have the following algebraic formulations:

Definitions

1.) The norm or length of a vector is obtained from the inner product as follows

$$\|u\| = \sqrt{(u,u)}$$

2.) Two vectors u and v are said to be orthogonal if $(u,v)=0$

3.) A basis

$$B = \{v_i : 1, 2, \ldots\}$$

is said to be orthogonal if

$$(v_i, v_j) = 0 \quad \text{when } i \neq j$$

4.) A normalized vector,

$$e_i = \frac{v_i}{\|v_i\|} \qquad i = 1, \ldots, n$$

is a unit vector. It has unit length:

$$\|e_i\| = \sqrt{\left(\frac{v_i}{\|v_i\|}, \frac{v_i}{\|v_i\|}\right)} = 1$$

5.) An orthogonal basis consisting of normalized vectors, $\{e_i = \frac{v_i}{\|v_i\|}\}$ is said to be an orthonormal basis if

$$(e_i, e_j) = \frac{(v_i, v_j)}{\|v_i\|\,\|v_j\|} = \left\{\begin{array}{ll} 0 & i \neq j \\ \frac{\|v_i\|^2}{\|v_j\|^2} = 1 & i = j \end{array}\right\} \equiv \delta_{ij}$$

6.) The algebraic expansion of a vector $\vec{u}$ relative to an orthonormal basis consists of the geometrical projections onto the orthogonal axes as follows:

Let

$$B = \{\vec{v}_1, \ldots\} \quad \text{be an othogonal basis}$$

The expansion

$$\vec{u} = \sum_i \vec{v}_i u^i$$

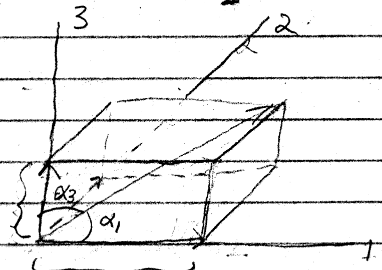
has expansion coefficients given by

$$(\vec{v}_j, \vec{u}) = \sum_i (\vec{v}_j, \vec{v}_i) u^i$$
$$= \|v_j\|^2 u^j \quad (\text{no sum})$$

Thus

$$\vec{u} = \sum_i \vec{v}_i \frac{(\vec{v}_i, \vec{u})}{\|\vec{v}_i\|^2} = \frac{\vec{v}_1}{\|\vec{v}_1\|}\left(\frac{\vec{v}_1}{\|\vec{v}_1\|}, \vec{u}\right) + \frac{\vec{v}_2}{\|\vec{v}_2\|}\left(\frac{\vec{v}_2}{\|\vec{v}_2\|}, u\right) + \cdots$$

$$\boxed{\begin{aligned} \vec{u} &= \vec{e}_1(\vec{e}_1, u) + \vec{e}_2(\vec{e}_2, \vec{u}) + \cdots \\ \vec{u} &= \underbrace{\vec{e}_1 \|\vec{u}\| \cos(e_1, u)}_{\text{orthogonal projection onto the } \vec{e}_1 \text{ axis}} + \vec{e}_2 \|\vec{u}\| \cos(e_2, u) + \cdots \end{aligned}}$$



$e_3(e_3, u)$

$e_1(e_1, u)$

Note: For a real vector space

$$(u, e_i) = (e_i, u) = \|u\| \cos \alpha_i$$

$$u \cdot u = \|u\|^2 \left[\cos^2\alpha_1 + \cos^2\alpha_2 + \cdots\right]$$

or

$$\left[\cos^2\alpha_1 + \cos^2\alpha_2 + \cdots\right] = 1$$

The sum of the squared <u>direction cosines</u> equals 1