LECTURE 25

Inner Product Spaces

I. Motivation

II. Definition

III. Examples

IV. Orthogonality and Normalization

In Gilbert Strang's "Linear Algebra & its Applications" (3rd Edition) Read Section 5.5 on complex matrices and complex inner products.

Inner Product Spaces.

I. Motivation

The feature common to all the vector space concepts introduced so far is that there was no commensurability along different dimensions. Different basis-induced coordinate lines would measure entirely different properties. Thus, the basis vector along the first coordinate line could measure the number of miles, the basis vector along the second the number of meters, the one along the third the number of light years, etc.

All this was before one introduced the concept of a metric (in LECTURE 9). There was no need for concepts such as the lengths of vectors or the angles between them.

The concept of a metric, is an entirely distinct ("geometric") structure. It is the Pythagorean theorem $a^2+b^2=c^2$, generalized from Euclidean to those vector spaces whose coordinate lines are observed to be related by virtue of their comensurability.

However, instead of restricting this generalization (to) real vector spaces, (Section 4.6 in Chapter 4 of J., R., & Arnold's 3rd Edition) we shall now do so to include also complex vector spaces. This is achieved by means of the following

II. Definition (Inner-Product Spaces)

An inner product on a complex vector space V is a bilinear, positive function such that for all vectors $u, v, w \in V$

a) $(u+v, w) = (u, w) + (v, w)$

b) $(u, cv) = c(u, v)$ $\forall$ complex number c

[Nota bene: the above is the physics/engineering convention. Some mathematics texts use the convention

$$(cu, v) = c(u, v)$$

instead.]

c) $(u, v) = \overline{(v, u)}$; (the bar denotes complex conjugation)

d) $(u, u) > 0$ whenever $u \neq 0$

III. Examples of inner products.

Ex. 1

For $V = R^n$ let $x = [x_1, \ldots, x_n]^T$

$$y = [y_1, \ldots, y_n]^T.$$

Then the standard inner product is

$$(x, y) = x_1 y_1 + \cdots + x_n y_n = [x_1, \ldots, x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \equiv x^T y$$

Ex. 2 a)

Let $V = R^2$

Then $(x, y) = x_1 y_1 - x_2 y_1 - x_1 y_2 + 4 x_2 y_2$

is a non-standard inner product

We do have positive-definiteness:

$$(x, x) = x_1^2 - 2x_1 x_2 + 4x_2^2 = (x_1 - x_2)^2 + 3x_2^2$$

Ex. 2b

Let $G = \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$; $V = R^2$;

then

$$(x, y) \equiv x^T G y = [x_1, x_2] \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= x_1 y_1 - x_2 y_1 - x_1 y_2 + 4 x_2 y_2$$

is an inner product.

Three comments:

(i) Property c) is necessary; otherwise one would have a contradiction with d):

$$(u,u)>0 \Rightarrow (iu,iu)=i(iu,u)=i(u,iu)=-(u,u)<0$$

(ii) $(cu,v)=\overline{(v,cu)}=\overline{c}\,\overline{(v,u)}=\overline{c}\,(u,v)$

Consequently,

b') $$\boxed{(cu,v)=\overline{c}(u,v)}$$

One says that (u,v) is anti-linear in the first factor, but is linear in the second factor.

(iii) Q: What is difference between a metric and an inner product?

A: A metric is more general in that

1.) $g(u,u)=0 \quad u\neq 0$

and

2.) $g(u,u)<0$

are allowed. However, a metric is also more special in that it presumes

1.) V is a real vector space, and

2.) $\dim V =$ finite.

Ex. 4 (Different inner products on the same V)
Let $V = P_2$; $p(x), q(x) \in P_2$

$$(p, q)_1 = p(0)q(0) + p(1)q(1) + p(2)q(2)$$

and

$$(p, q)_2 = \int_0^1 p(t)\,q(t)\,dt$$

are two different inner products

Ex. 5
Let V = set of continuous complex functions on $[a, b]$

$$(f, g) = \int_a^b \bar{f}(t)\,g(t)\,dt$$

IV. Orthogonality and Normalization are geometrical concepts which have the following algebraic formulations:

Definitions

1.) The <u>norm</u> or <u>length</u> of a vector is obtained from the inner product as follows

$$\|u\| = \sqrt{(u, u)}$$

2.) Two vectors u and v are said to be <u>orthogonal</u> if $(u, v) = 0$

3.) A basis

$$B = \{v_i : 1, 2, \ldots\}$$

is said to be <u>orthogonal</u> if

$$(v_i, v_j) = 0 \qquad \text{when } i \neq j$$

4.) A <u>normalized</u> vector,

$$e_i = \frac{v_i}{\|v_i\|} \qquad i = 1, \ldots, n$$

is a <u>unit</u> vector. It has unit length:

$$\|e_i\| = \sqrt{\left(\frac{v_i}{\|v_i\|}, \frac{v_i}{\|v_i\|}\right)} = 1$$

5.) An orthoganal basis consisting of normalized vectors, $\{ e_i = \frac{v_i}{\|v_i\|} \}$ is said to be an orthonormal basis if

$$(e_i, e_j) = \frac{(v_i, v_j)}{\|v_i\| \, \|v_j\|} = \left\{ \begin{array}{ll} 0 & i \neq j \\ \frac{\|v_i\|^2}{\|v_j\|^2} = 1 & i = j \end{array} \right\} \equiv \delta_{ij}$$

6.) The algebraic expansion of a vector $\vec{u}$ relative to an orthonormal basis consists of the geometrical projections onto the orthogonal axes as follows;

Let

$B = \{ \vec{v}_1, \ldots \}$ be an othogonal basis

The expansion

$$\vec{u} = \sum_i \vec{v}_i u^i$$

has expansion coefficients given by

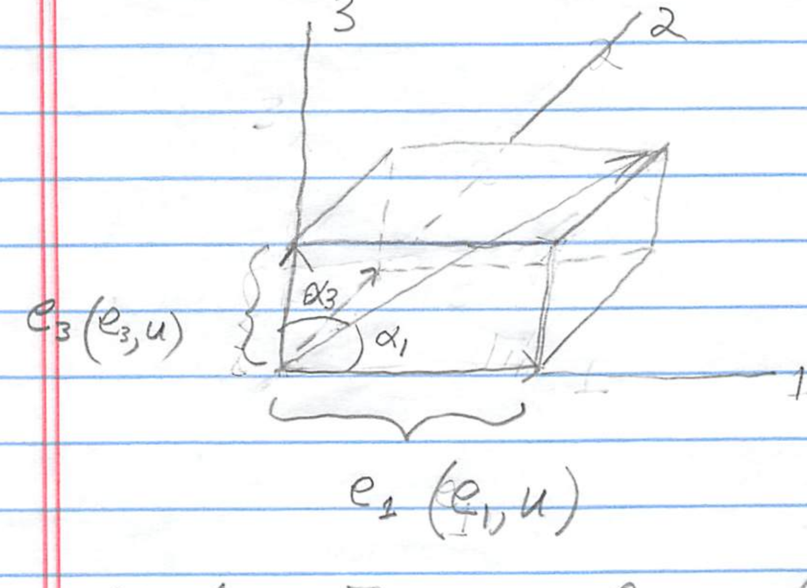
$$(\vec{v}_j, \vec{u}) = \sum_i (\vec{v}_j, \vec{v}_i) u^i$$

$$= \|v_j\|^2 u^j \quad (\text{no sum})$$

Thus

$$\vec{u} = \sum_i \vec{v}_i \frac{(v_i, \vec{u})}{\|v_i\|^2} = \frac{\vec{v}_1}{\|v_1\|}\left(\frac{\vec{v}_1}{\|v_1\|}, \vec{u}\right) + \frac{\vec{v}_2}{\|v_2\|}\left(\frac{\vec{v}_2}{\|v_2\|}, u\right) + \cdots$$

$$\boxed{\begin{aligned} \vec{u} &= \vec{e}_1(\vec{e}_1, u) + \vec{e}_2(\vec{e}_2, \vec{u}) + \cdots \\ \vec{u} &= \underbrace{\vec{e}_1 \|\vec{u}\| \cos(e_1, u)}_{\substack{\text{orthogonal} \\ \text{projection} \\ \text{onto the } \vec{e}_1 \text{ axis}}} + \vec{e}_2 \|\vec{u}\| \cos(e_2, u) + \cdots \end{aligned}}$$



Note: For a real vector space

$$(u, e_i) = (e_i, u) = \|u\| \cos \alpha_i$$

$$u \cdot u = \|u\|^2 \left[\cos^2\alpha_1 + \cos^2\alpha_2 + \cdots\right]$$

or

$$\left[\cos^2\alpha_1 + \cos^2\alpha_2 + \cdots\right] = 1$$

The sum of the squared <u>direction</u> cosines equals 1