LECTURE 27

1. Orthogonal (= Orthonormal) Matrices

2. Right vs. Left Inverses

3. Application to Orthogonal Matrices

4. Rigid Transformation,

I. ORTHOGONAL (= Orthonormal) MATRICES

a) The application of the G-S orthogonalition process to <u>any rectangular matrix</u> A with <u>independent columns</u> yields the result that A can be factored into

$$A = QR$$

where the columns of Q are orthogonal and normalized and R is an upper triangular, invertible square matrix.

b) If A is square, then so is Q, and Q becomes an "orthogonal" matrix. Thus one has the following

<u>Definition</u>

An <u>orthogonal</u> matrix is a square matrix with orthonormal columns. It has the property

$$\begin{bmatrix} \text{---}\; e_1^T \;\text{---} \\ \text{---}\; e_2^T \;\text{---} \\ \vdots \\ \text{---}\; e_n^T \;\text{---} \end{bmatrix} \begin{bmatrix} \vert & \vert & & \vert \\ e_1 & e_2 & \cdots & e_n \\ \vert & \vert & & \vert \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \end{bmatrix}$$

because

$$\langle e_i | e_j \rangle = \delta_{ij} .$$

For orthogonal matrices one has therefore

$$\boxed{Q^T \; Q = I} \qquad (*)$$

where Q^T is the transpose of Q:

$$\left[Q_{ij}\right]^T = \left[Q_{ji}\right].$$

Thus Eq. (*) implies

$$\boxed{Q^T = Q^{-1}}$$ (= "left" inverse, to be precise)

II. RIGHT and LEFT INVERSE of a MATRIX

A matrix with o.n. columns satisfies

$$Q^T Q = \begin{bmatrix} \cdots e_1 \cdots \\ \vdots \\ \cdots e_k \cdots \end{bmatrix} \begin{bmatrix} \vdots & & \vdots \\ e_1 & \cdots & e_k \\ \vdots & & \vdots \end{bmatrix}$$

Such a matrix has a left inverse

$Q_L^{-1} = Q^T$: $\quad Q_L^{-1} Q = I$

Question: What about $QQ_R^{-1} = I$?

Answer: **Theorem 27.1**: Let $Q: U \xrightarrow{Q} V$

a) Q is 1-1 $\Leftrightarrow$ Q has a left inverse:
$\exists\, Q_L^{-1}$ such that $Q_L^{-1} Q = I_U$

b) Q is onto $\Leftrightarrow$ Q has a right inverse:
$\exists\, Q_R^{-1}$ such that $QQ_R^{-1} = I_V$

proof: $\Leftarrow$

a) Consider $Qx_1 = Qx_2$

Then $\underbrace{Q_L^{-1} Q}_{I_U} x_1 = \underbrace{Q_L^{-1} Q}_{I_U} x_2$

$\therefore\ x_1 = x_2$, ie Q is 1-1.

b) Consider $y \in V$ and $QQ_R^{-1} y = y$

Let $x = Q_R^{-1} y$. Then $Qx = y$. Thus Q is onto.

Comment:
The proof for $\Rightarrow$ is more delicate

Example 1

$Q: R^3 \to R^2$: $QQ_R^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ on R^2

Q is onto
Q is not 1-1

Example 2

$Q: R^2 \to R^3$ $\quad Q_L^{-1} Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ on R^2

Q is 1-1
Q not onto.

Corollary 27.1 (1-1 and onto $\Rightarrow Q_L^{-1} = Q_R^{-1}$)

Let $Q: U \to V$
Q has a left inverse Q_L^{-1} : $Q_L^{-1} Q = I_U$
Q has a right inverse Q_R^{-1} : $QQ_R^{-1} = I_V$

Conclusion:

$$Q_L^{-1} = Q_R^{-1} \quad (\equiv Q^{-1})$$

proof: $Q_R^{-1} = I_U Q_R^{-1} = (Q_L^{-1} Q) Q_R^{-1} = Q_L^{-1}(QQ_R^{-1})$
$= Q_L^{-1} I_V$
$= Q_L^{-1}$.

III APPLICATION to ORTHOGONAL MATRICES

For an orthogonal matrix $Q: V \to V$, it columns form a basis for V. Thus $R(Q) = V$. This means that Q is onto. Consequently, one has

Proposition 27.1: Being onto, an $n \times n$ orthogonal matrix has a right inverse: $Q Q_R^{-1} = I$

III, Corollary 27.2 $\boxed{Q Q^T = I = Q^T Q}$

proof: $Q_R^{-1} = I Q_R^{-1} = (Q^T Q) Q_R^{-1} = Q^T$

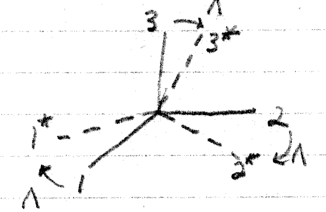
Thus $I = Q Q_R^{-1} = Q Q^T$

Comment: This is a remarkable result. It says that if the columns are o.n. then the rows are o.n. also, even though they point into entirely different directions.

IV RIGID TRANSFORMATION

Definition 27.2: A rigid transform $\Lambda: V \to V$ is one which transforms an o.n. basis into an o.n. basis.

$$\{e_1, \dots, e_n\} = \overset{\text{old}}{\text{o.n. basis}} \xrightarrow{\Lambda} \{e_1^*, \dots, e_n^*\} = \overset{\text{new}}{\text{o.n. basis}}$$



Proposition 27.2: A rigid transformation is orthogonal.

Let Λ: 1. $e_i^* = \Lambda(e_i) = e_j \Lambda_{ji}$

2. The matrix elements Λ_{ji} are computed using the orthonormality

$$\boxed{e_k \cdot e_i^* = \underbrace{e_k \cdot e_j}_{\delta_{kj}} \Lambda_{ji} = \Lambda_{ki}}$$

3. Thus

$$\delta_{\ell i} = e_\ell^* \cdot e_i^* = e_k \Lambda_{k\ell} \cdot e_j \Lambda_{ji}$$

$$= \Lambda_{k\ell} \Lambda_{ki} \quad \text{"}\ell^{th} \text{ column} \times i^{th} \text{ col."}$$

$$= \Lambda^T_{\ell k} \Lambda_{ki}$$

$$\delta_{\ell i} = (\Lambda^T \Lambda)_{\ell i} \quad \text{"index notation"}$$

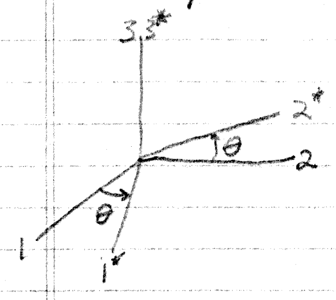
$$\therefore \boxed{I = \Lambda^T \Lambda} \quad \text{"matrix notation"}$$

Comment: The matrix element Λ_{ki} can be computed using orthonormality

$$\boxed{e_k \cdot e^*_i = e_k \cdot e_j \Lambda_{ji} = \Lambda_{ki} = \cos(e_k, e^*_i)}$$

$$\boxed{= \text{cos of } \measuredangle \text{ between old } k\text{-axis and new } i\text{-axis.}}$$

Example 1: Rotation by θ around z axis



$$[\Lambda_{ki}] = [e_k \cdot e^*_i]$$

$$= \begin{bmatrix} \cos\theta & \cos(\frac{\pi}{2}+\theta) & \cos\frac{\pi}{2} \\ \cos(\frac{\pi}{2}-\theta) & \cos\theta & \cos\frac{\pi}{2} \\ \cos\frac{\pi}{2} & \cos\frac{\pi}{2} & \cos 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Active view point:

1.) Rotate the vector using Λ:

$X \rightsquigarrow X^* = \Lambda(X)$.

$\vec{e}_i X^i \rightsquigarrow \Lambda(e_i X^i) = \Lambda(e_i) X^i$

$= e^*_i X^i$ where $e^*_i = e_j \Lambda_{ji}$

$=$ rotated vector

Passive view point:

2.) Change basis but vector stays fixed:

Let $\vec{e}_i X^i = \vec{X} = e^*_k X^{*k}$ same vector

or $\vec{e}_j X^j = e_j \Lambda_{jk} X^{*k}$ but

$X^j = \Lambda_{jk} X^{*k}$ different rep'n.

$[X]_{old} = \Lambda [X]_{new}$