LECTURE 28

I. Rigid Transformation

II. "Active" vs. "Passive" view-point.

III. Orthogonal Projections

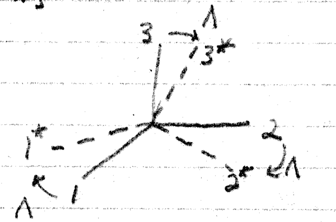
V. Least Squares Problem

VI. Least Squares Approximation

I. RIGID TRANSFORMATION

Definition 28.1: A rigid transform $\Lambda: V \to V$ is one which transforms an o.n. basis into an o.n. basis

$$\{e_1, \ldots, e_n\} = \overset{\text{old}}{\text{o.n. basis}} \xrightarrow{\Lambda} \{e_1^*, \ldots, e_n^*\} = \overset{\text{new}}{\text{o.n. basis}}$$



Proposition 28.1: A rigid transformation is orthogonal.

Let Λ: 1. $e_i^* = \Lambda(e_i) = e_j \Lambda_{ji}$

2. The matrix elements Λ_{ji} are computed using the orthonormality

$$\boxed{e_k \cdot e_i^* = \underbrace{e_k \cdot e_j}_{\delta_{kj}} \Lambda_{ji} = \Lambda_{ki}}$$

3. Thus

$$\delta_{\ell i} = e_\ell^* \cdot e_i^* = e_k \Lambda_{k\ell} \cdot e_j \Lambda_{ji}$$

$$= \Lambda_{k\ell} \Lambda_{ki} \quad \text{"}\ell^{th}\text{ column} \times i^{th}\text{ col."}$$

$$= \Lambda^T_{\ell k} \Lambda_{ki} \quad \text{"index notation"}$$

$$\delta_{\ell i} = (\Lambda^T \Lambda)_{\ell i}$$

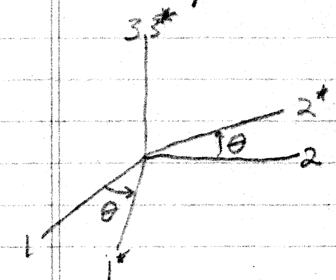
$$\therefore \boxed{I = \Lambda^T \Lambda} \quad \text{"matrix notation"}$$

Comment: The matrix element Λ_{ki} can be computed using orthonormality

$$\boxed{e_k \cdot e_i^* = e_k \cdot e_j \Lambda_{ji} = \Lambda_{ki} = \cos(e_k, e_i^*)}$$

$$\boxed{= \cos \text{ of } \measuredangle \text{ between old } k\text{-axis and new } i\text{-axis.}}$$

Example 1: Rotation by θ around z axis



$$[\Lambda_{ki}] = [e_k \cdot e_i^*]$$

$$= \begin{bmatrix} \cos\theta & \cos(\frac{\pi}{2}+\theta) & \cos\frac{\pi}{2} \\ \cos(\frac{\pi}{2}-\theta) & \cos\theta & \cos\frac{\pi}{2} \\ \cos\frac{\pi}{2} & \cos\frac{\pi}{2} & \cos 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

active view point

1.) Rotate the vector using Λ:

$X \rightsquigarrow X^* = \Lambda(X)$.

SKIP THIS

$$\vec{e}_i x^i \rightsquigarrow \Lambda(e_i x^i) = \Lambda(e_i) x^i$$
$$= e_i^* x^i \quad \text{where } e_i^* = e_j \Lambda_{ji}$$
$$= \text{rotated vector} \quad (\text{different from } X)$$

Passive view point

2.) change basis but vector stays fixed:

Let $\vec{e}_i x^i = \vec{X} = e_k^* x^{*k}$ — same vector,

or $\vec{e}_j x^j = e_j \Lambda_{jk} x^{*k}$ — but

$$x^j = \Lambda_{jk} x^{*k}$$
$$[X]_{old} = \Lambda [X]_{new}$$

different rep'n.

II. ACTIVE vs PASSIVE VIEWPOINT

A rotation Λ can be used for two different purposes:

1. Rotate a vector using Λ:

$$\vec{x} \rightsquigarrow \vec{X}^* = \Lambda(\vec{X})$$

$$\begin{aligned}\vec{e}_i x^i \rightsquigarrow \Lambda(\vec{e}_i x^i) &= \Lambda(\vec{e}_i) x^i \\ &= (\vec{e}_j \Lambda_{ji}) x^i \\ &= \vec{e}_j \underbrace{(\Lambda_{ji} x^i)} \\ \vec{X}^* = \vec{e}_j\, x^{*j} &= \text{rotated vector}\end{aligned}$$

Thus we have a different vector
but the basis is the same,
i.e. the orientation of the basis vectors
remains fixed relative to the fixed stars
while the vector rotates
In physics and engineering, this use of Λ is called the active viewpoint,

2. Change the basis, but keep the vector fixed (relative to the fixed stars):

$$\vec{e}_i x^i = \vec{X} = \vec{e}^*_k x^{*k} \quad \text{(same vector different basis)}$$

$$\vec{e}_i x^i = \vec{e}_i \Lambda_{ik} x^{*k}$$

Thus

$$x^i = \Lambda_{ik} x^{*k}$$

This expresses a transition between two representations whose transition matrix (see page 19.5) is

$$P_{\ell i} = \Lambda^{-1}_{\ell i}$$

(Indeed, using $\Lambda^{-1}_{\ell i} \Lambda_{ik} = \delta_{\ell k}$, one obtains $x^{*\ell} = \Lambda^{-1}_{\ell i} x^i$, the transition from the old $[x^i]$ to the new $[x^{*\ell}]$ representation of $\vec{X}$.)

Using Λ for this purpose is called the passive viewpoint. Thus one has
the active viewpoint, where one
rotates the vector $\vec{X}$ while keeping the

basis fixed (relative to the fixedstars) vs the passive viewpoint where one keeps the vector fixed (relative to the fixedstars) while changing the reference frame by means of a rotation.

III. Orthogonal Projections

The Gram-Schmidt process consisted of removing the orthogonal projections

$$\frac{\langle u_j | v \rangle}{\|u_j\|^2} \qquad j = 1, \ldots, N$$

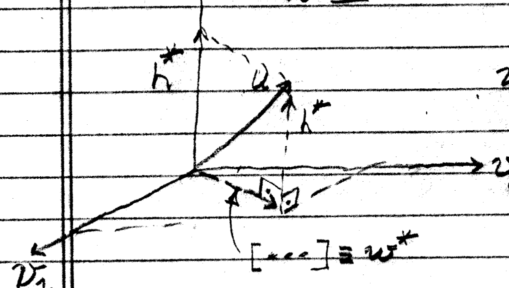
from v,

$$h^* = v - \underbrace{\left[\frac{u_1 \langle u_1 | v \rangle}{\|u_1\|^2} + \cdots + \frac{u_N \langle u_N | v \rangle}{\|u_N\|^2} \right]}_{w^*}$$

Here $[\cdots] \equiv w^*$ is the orthogonal projection of v onto the subspace $Sp\left(\{u_j\}_{j=1}^N\right) \equiv W_N$

Removing these projections resulted in h^*, a vector which was orthogonal to: W_N

$$h^* \perp W \ : \ \langle u_j | h^* \rangle = 0; \ j = 1, \ldots, N,$$



whenever the vectors $u_1, \ldots, u_N$ where mutually orthogonal.

IV. LEAST SQARES PROBLEM

One of the consequences of this orthogona- (the observation) lity is that h^* minimizes the squared distance of v from W_N. In other words, $\exists$ an optimal $w^* \in W_N$ such that

$$\|h^*\|^2 \leq \|v - w\|^2 \quad \forall \, w \in W_N.$$

This observation motivates the following the following orthogonal projection problem:

Least-Squares Problem

Let V be an inner product space

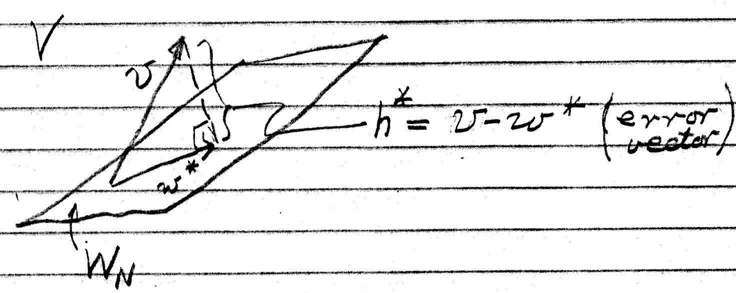
Let $W_N \subset V$ be a subspace.

GIVEN $v \in V$, FIND in W_N a vector w^* such that

$$\|v - w^*\| = \min_{w \in W_N} \|v - w\|$$

i.e.

$$\|v - w^*\| \le \|v - w\| \qquad \forall w \in W_N$$



Comment:
w^* is called (i) the orthogonal projection of v onto W_N or

(ii) nearest vector in W_N to v

(iii) the <u>least squares approximation</u> to v.

V. SOLUTION TO LEAST SQUARES PROBLEM.

This problem is solved by the following

Theorem (Perpendicular distance = Minimum distance)

Given:

V is an inner product space

$W_N \subset V$ is a subspace of V

Conclusion

$\exists$ a unique $w^* \in W_N$ such that

$$\langle v - w^* | w \rangle = 0 \qquad \forall\, w \in W_N$$

if and only if ($\Longleftrightarrow$)

$$\left\{ \|v-w\|^2 \equiv \mathcal{E}^2(w) \text{ has a minimum at } w = w^* \right\}$$

or equivalently, $\exists\, w^* \in W_N$ such that

$$\|v - w^*\| \le \|v - w\| \qquad \forall\, w \in W_N$$

Proof ($\Rightarrow$):

Let $w \in W_N$. Then

$$\|v-w\|^2 = \langle v - w^* + w^* - w, v - w^* + w^* - w \rangle$$

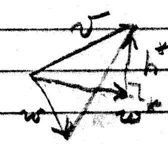
$$= \|v-w^*\|^2 + \underbrace{\langle w^* - w, v - w^* \rangle}_{0} + \underbrace{\langle v - w^*, w^* - w \rangle}_{0}$$

$$+ \|w^* - w\|^2$$

because $w^* - w \in W_N$

Thus

$$\|v-w^*\|^2 \le \|v-w\|^2 \qquad \forall\, w \in W_N$$



Comment

$$\|v-w\|^2 = \overbrace{\|v-w^*\|}^{h^*}{}^2 + \|w^*-w\|^2 \text{ is}$$

the Pythagorean theorem