LECTURE 28

I. Rigid Transformations

II. "Active" vs. "Passive" view point.

III. Orthogonal Projections

IV. Least Squares Problem

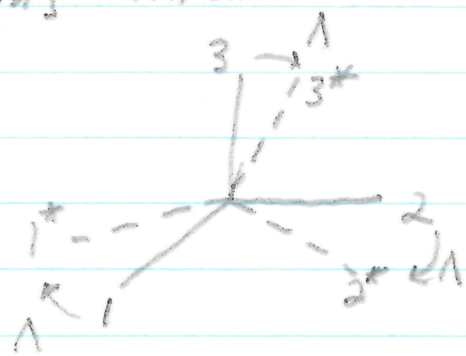
[V. Least Squares Approximation]

Read 4.6 in JR&A 3rd Ed.
3.1-3.4 in G. Strang 3rd Ed.

I. RIGID TRANSFORMATIONS

Definition 28.1: A rigid transform $\Lambda : V \to V$ is one which transforms an o.n. basis into an o.n. basis.

$$\underbrace{\{e_1, \ldots, e_n\}}^{\text{old}} = \text{o.n. basis} \xrightarrow{\Lambda} \{e_1^*, \ldots, e_n^*\} = \text{o.n. basis}^{\text{new}}$$



Proposition 28.1: A rigid transformation is orthogonal.

Let Λ: 1. $e_i^* = \Lambda(e_i) = e_j \Lambda_{ji}$

2. The matrix elements Λ_{ji} are computed using the orthonormality

$$\boxed{e_k \cdot e_i^* = \underbrace{e_k \cdot e_j}_{\delta_{kj}} \Lambda_{ji} = \Lambda_{ki}}$$

3. Thus

$$\delta_{\ell i} = e_\ell^* \cdot e_i^* = e_k \Lambda_{k\ell} \cdot e_j \Lambda_{ji}$$

$\Lambda_{11} \quad \Lambda_{12}$

$\Lambda_{n1} \quad \Lambda_{n2}$

$$
\left.
\begin{aligned}
&= \Lambda_{k\ell} \Lambda_{ki} \\
&= \Lambda^T_{\ell k} \Lambda_{ki} \\
\delta_{\ell i} &= (\Lambda^T \Lambda)_{\ell i}
\end{aligned}
\right\}
$$

"ℓ^{th} column $\times$ i^{th} col."

"index notation"

$$\therefore \boxed{I = \Lambda^T \Lambda}$$

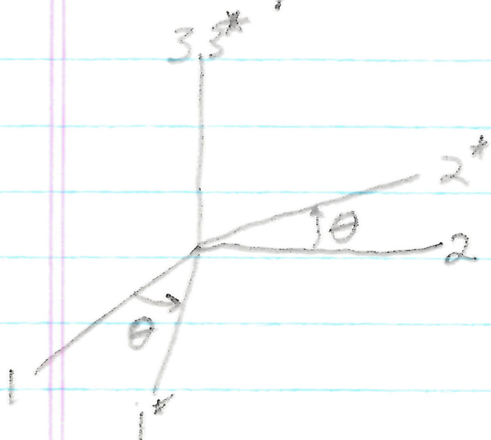
"matrix notation"

Comment: The matrix element Λ_{ki} can be computed using orthonormality

$$\boxed{e_k \cdot e_i^* = e_k \cdot e_j \Lambda_{ji} = \Lambda_{ki} = \cos(e_k, e_i^*)}$$

$$\boxed{= \text{cos of } \measuredangle \text{ between old } k\text{-axis and new } i\text{-axis.}}$$

Example 1: Rotation by θ around z axis



$$[\Lambda_{ki}] = [e_k \cdot e_i^*]$$

$$= \begin{bmatrix} \cos\theta & \cos(\frac{\pi}{2}+\theta) & \cos\frac{\pi}{2} \\ \cos(\frac{\pi}{2}-\theta) & \cos\theta & \cos\frac{\pi}{2} \\ \cos\frac{\pi}{2} & \cos\frac{\pi}{2} & \cos 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

active view point

1.) Rotate the vector using Λ:

$x \rightsquigarrow x^* = \Lambda(x)$.

$\vec{e}_i x^i \rightsquigarrow \Lambda(e_i x^i) = \Lambda(e_i) x^i$ SKIP THIS

$= e_i^* x^i$ where $e_i^* = e_j \Lambda_{ji}$

= rotated vector (different from x)

2.) Change basis but vector stays fixed:

Passive view point

Let $\vec{e}_i x^i = \vec{x} = e_k^* x^{*k}$ same vector

or $\vec{e}_j x^j = e_j \Lambda_{jk} x^{*k}$ but

$x^j = \Lambda_{jk} x^{*k}$ different rep'n.

$[x]_{old} = \Lambda [x]_{new}$

II. ACTIVE vs PASSIVE VIEWPOINT.

A rotation Λ can be used for two different purposes:

1. Rotate a vector using Λ:

$$\vec{X} \rightsquigarrow \vec{X}^* = \Lambda(\vec{X})$$

$$\vec{e}_i X^i \rightsquigarrow \Lambda(\vec{e}_i X^i) = \Lambda(\vec{e}_i) X^i$$
$$= (\vec{e}_j \Lambda^j{}_i) X^i$$
$$= \vec{e}_j \underbrace{(\Lambda^j{}_i X^i)}$$
$$\vec{X}^* = \vec{e}_j X^{*j} = \text{rotated vector}$$

same basis

Thus we have a different vector but the basis is the same, i.e. the orientation of the basis vectors remains fixed relative to the fixed stars while the vector rotates.

In physics and engineering this use of Λ is called the active viewpoint.

2. Change the basis, but keep the vector fixed (relative to the fixed stars):

$$\vec{e}_i x^i = \vec{X} = e^*_k x^{*k} \quad \text{(\underline{same} vector different basis)}$$

$$\vec{e}_i x^i = \vec{e}_i \Lambda_{ik} X^{*k}$$

Thus

$$x^i = \Lambda_{ik} X^{*k} \qquad x^{*i} = \underbrace{P_{ik}}_{\Lambda^{-1}_{ik}} X^k$$

$$[\vec{X}]_{new} = P_{new\leftarrow old}[\vec{X}]_{old}; \; X^{*i} = \Lambda^{-1}{}_{ik} X^k = P_{ik} X^k \qquad \Lambda^{-1}_{ik} (= \Lambda_{ki})$$

This expresses a transition between two representations whose transition matrix (see page 19.5) is

$$P_{\ell i} = \Lambda^{-1}{}_{\ell i}$$

(Indeed, using $\Lambda^{-1}{}_{\ell i}\Lambda_{ik} = \delta_{\ell k}$, one obtains $x^{*\ell} = \Lambda^{-1}{}_{\ell i} x^i$, the transition from the old $[x^i]$ to the new $[x^{*\ell}]$ representation of $\vec{x}$.)

Using Λ for this purpose is called the passive viewpoint. Thus one has the active viewpoint, where one rotates the vector $\vec{x}$ while keeping the basis fixed (relative to the fixed stars) vs the passive viewpoint where one keeps the vector fixed (relative to the fixed stars) while changing the reference frame by means of a rotation.

III. Orthogonal Projections

The Gram-Schmidt process consisted of removing the orthogonal projections

$$\frac{\langle u_j | v \rangle}{\|u_j\|^2} \qquad j = 1, \ldots, N$$

from v,

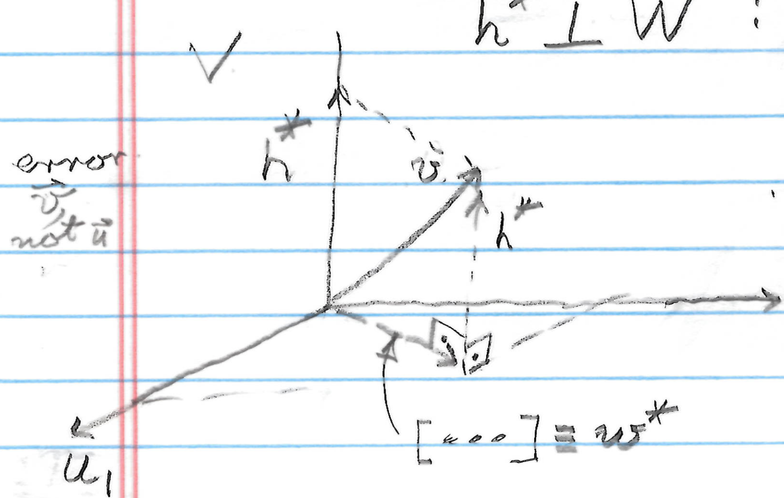
$$h^* \equiv v - \underbrace{\left[\frac{u_1 \langle u_1 | v \rangle}{\|u_1\|^2} + \cdots + \frac{u_N \langle u_N | v \rangle}{\|u_N\|^2} \right]}_{w^*}$$

Here $[\cdots] \equiv w^*$ is the orthogonal projection of v onto the subspace $Sp\left(\{u_j\}_{j=1}^{N}\right) \equiv W_N$

Removing these projections resulted in h^*, a vector which was orthogonal to: W_N

$$h^* \perp W \; : \; \langle u_j | h^* \rangle = 0 \, ; \; j = 1, \ldots, N,$$

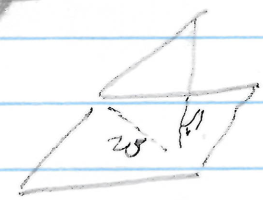
whenever the vectors $u_1, \ldots, u_N$ where mutually orthogonal.



IV. LEAST SQARES PROBLEM

One of the consequences of this orthogonality is the observation that h^* minimizes the squared distance of v from W_N. In other words, $\exists$ an optimal $w^* \in W_N$ such that

$$\|h^*\|^2 \leq \|v - w\|^2 \quad \forall w \in W_N.$$



This observation motivates the following orthogonal projection problem:

Least-Squares Problem

Let V be an inner product space

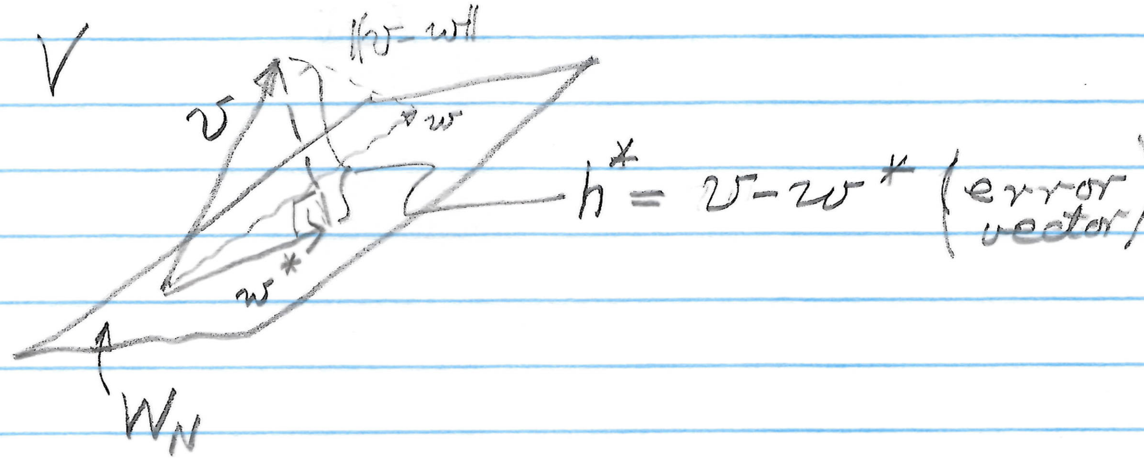
Let $W_N \subset V$ be a subspace.

GIVEN $v \in V$, FIND in W_N a vector w^* such that

$$\|v - w^*\| = \min_{w \in W_N} \|v - w\|$$

i.e.

$$\|v - w^*\| \le \|v - w\| \qquad \forall\, w \in W_N$$



Comment:
w^* is called (i) the orthogonal projection of v onto W_N or

(ii) nearest vector in W_N to v

(iii) the <u>least squares approximation</u> to v.