LECTURE 2

Friday

Vector Spaces: Their observational Basis.

The Subspace Theorem

Spanning set

For LECTURE 3 $\left\{ \begin{array}{l} \text{Linear independence} \\ \text{Basis; coordinates} \\ \text{Basis-induced isomorphism} \end{array} \right.$

Vector Spaces: Their Basis in Observation

Reconsider the vector space consisting of the set of inventories of a supermarket. (*Nota bene:* In the science of supply chain management, an inventory with a negative number of items is called a "backlog" or "backorder".) But this time consider the inventories consisting of fruits <u>and</u> vegetables (e.g. cucumbers, tomatoes, asparagus).

In this case one has fruit-plus-vegetable inventories. Each one is a composite inventory which consists of a fruit inventory plus a vegetable inventory. This observation is mathematized by the equation:

$$\vec{v} = \overrightarrow{\text{fruit inventory}} + \overrightarrow{\text{vegetable inventory}} \tag{1}$$

$$\equiv \vec{v_f} + \overrightarrow{v_{veg}} \tag{2}$$

The set of such vectors is a new vector space, the space of fruit-plus-vegetable inventories, which is formed from the vector spaces V_f and V_{veg} and which is designated by

$$V = \{\overrightarrow{\text{fruit inventory}}\} \oplus \{\overrightarrow{\text{vegetable inventory}}\}$$

$$\equiv V_f \oplus V_{veg}$$

In the ensuing lecture we shall identify V_f and V_{veg} as *subspaces* of V. Furthermore, the vector space V is called the *direct sum* of V_f and V_{veg}. It is called a *direct* sum, because the only inventory common to both is the trivial inventory, i.e. the zero vector:

$$V_f \cap V_{veg} = \{\vec{0}\}$$

Whenever that is the case, the decomposition Eq. (1) is necessarily *unique.* The validity of this uniqueness claim is highlighted by the question

Q: Why does this hold for *every* inventory in V? and its answer, an observation about the nature of things:

A: Everything which exists has a specific nature:

(a) a fruit is a fruit and a vegetable is a vegetable.

(b) More generally, *this* is *this* and *that* is *that,* i.e. A is A.

(c) If it is a fruit it is *not* a non-fruit, i.e. it is not a vegetable; if it is a vegetable, it is *not* a non-vegetable, i.e. it is not a fruit.

This is Aristotle's Law of Identity in action. It is a conceptualized observation about the nature of things. It is a pre-condition for any type of valid reasoning, inductive or deductive, in science, in engineering, in mathematics, in the humanities, indeed – in all of knowledge.

Stated negatively, this law says:
Contradictions do not exist in the physical world; the existence of a contradiction is *prima facia* evidence for erroneous thinking.

I. The Subspace Theorems

Q: How does one determine whether a set of objects forms a vector space?

A: This task is accomplished by verifying that they obey properties 3) a-d and 4) a-d of Definition 1 on pages 1.7-1.8 in Lecture 1.

However, under certain circumstances this 8-step task can be reduced to one of two step only. The key to success lies in identifying this set of objects as a subset of some vector space, and then merely checking closure under addition and multiplication by scalars.

One starts with

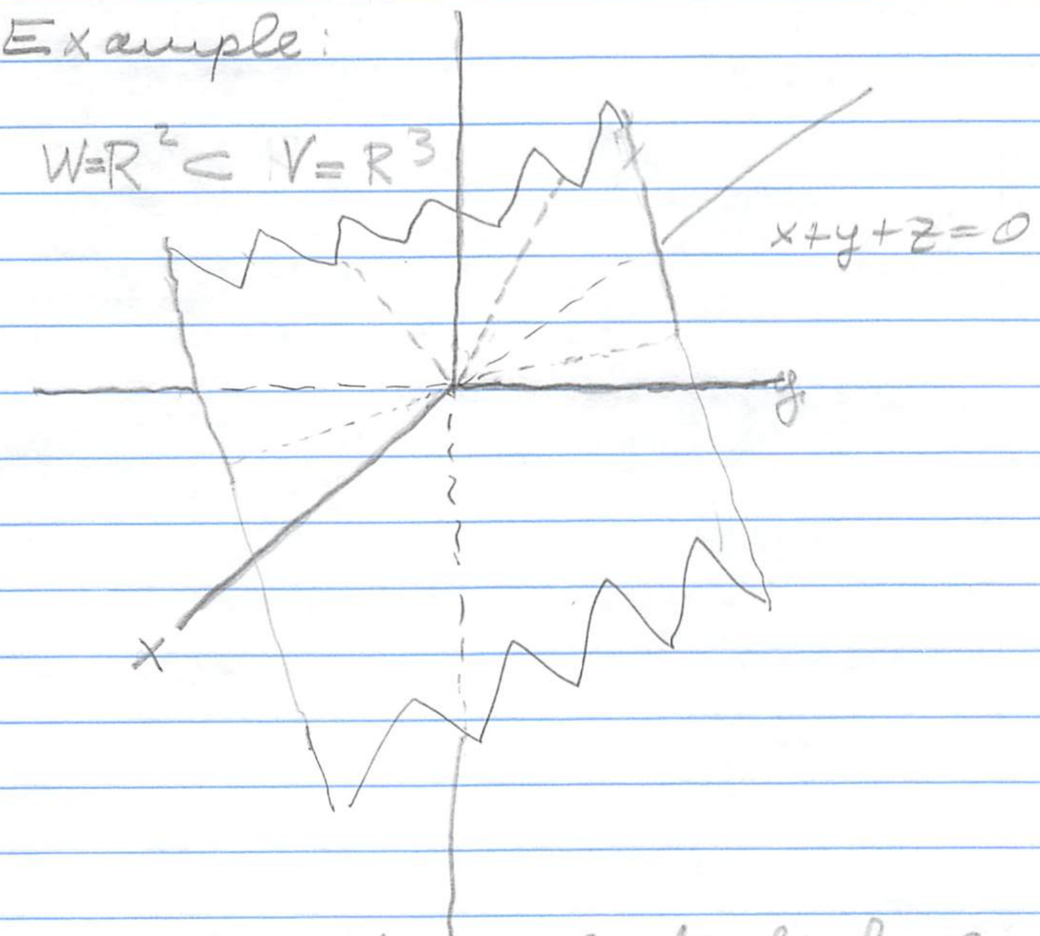
Definition 2 (subspace)

Let V, W both be vector spaces

$W \subset V$, i.e W is a subset of V

Then W is called a subspace of V.

Example:



So a subspace is a subset which also is a vector space.

Sub(vector)spaces are easy to recognize

because

Theorem 1

Let a) $W \subset V$ = vector space
b) W is non-empty
i.e. W is a non-empty subset of V

Conclusion

$\boxed{W \text{ is a subspace}} \Leftrightarrow \left\{ \begin{array}{l} (1)\ u+v \in W \\ (2)\ cu \in W \end{array} \right\}$ $\begin{array}{l} u, v \in W \\ c \in F \end{array}$

Comment:
More compactly, we have

W is a subspace $\Leftrightarrow cu+v \in W$

Proof: $\Rightarrow$ is obvious because W is a vector space.

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~~$\Leftarrow$ 1. commutativity, associativity, and distributivity are inherited from V~~

~~2. W contains the zero vector because $W \ni 0 \cdot \vec{u} = \vec{0}$ as shown in Lecture 1.~~

$\Leftarrow$ 1. commutativity 3a
associativity 3b
distributivity 4c, 4d
$1\vec{u} = \vec{u}$ 4a
$(c_1 c_2)\vec{u} = c_1(c_2\vec{u})$ 4b

inherited from V

2. W contains the zero vector (3c):
W is non-empty $\Rightarrow \exists \vec{u} \in W \cap V$. W is closed under scalar mpl'n $\Rightarrow 0\cdot\vec{u} \in W$. $0\cdot\vec{u} = \vec{0} \Rightarrow \vec{0} \in W$.

(given) Shown in Lecture 1

3. For any $u \in W$, W contains the additive inverse (3d):

Let $u \in W$. Then $(-1)\vec{u} = -\vec{u} \in W$

given homework problem 1

Comment:
Theorem 1 simplifies the task of determining whether or not W is a vector space: all one needs to do is verify $c\vec{u} + \vec{v} \in W$ whenever $\vec{u}, \vec{v} \in W$

Example.

For $m < n$

$$P_m = \{ p(x) = a_m x^m + \cdots + a_1 x + a_0 : a_m, \ldots, a_0 \in R \}$$

is a subspace of P_n. Indeed,

a) we know that P_n is a vector space.
b) " " " $P_m \subset P_n$.
Thus
c) $\boxed{P_m \text{ is a subspace}}$: because P_m is closed under addition & mplication.

~~II Spanning Sets~~

~~We shall now give several defintions, and from these deduce several theorems.~~

~~The difficult part of this is to come up with the definitions; this is basically an inductive process whose hard work we owe to the 18th and 19th century mat~~

II Spanning Sets

We shall now state several definitions and from them deduce several theorems.

A definition states the distinguishing property(ies) of a concept ["span of Q"; "spanning set for V"]. The difficult part of this enterprise is to form the concept and to identify its distinguishing (i.e. essential) property. This is largely done by a process of inductive reasoning, which is much more difficult than deductive reasoning, which is the formulation of the theorems.

In other words, arriving at these theorems depends crucially on the concepts identified by means of their definition. The quality and validity of the theorems depends on the quality and validity of the concepts which make the theorems possible.

The formation of concepts are green lights for the statements of theorems, relations between the concepts.

Comment and Outlook:

In the first lecture we considered vectors wholesale
In this lecture we shall consider them retail.
In the first lecture we where interested in recognizing whether the whole set is indeed a vector space and how one does this.

In this lecture we will single out finite number of vectors and use them to coordinatize the vector space. It turns out that there is an enourmous amount of flexibility in establishing a coordinate system, and This flexibility is a reflection of.

the flexibility in performing measurements

for identifying the nature of things in

the physical world.

II. Spanning Set

A spanning set is a set of vectors, which, given any vector $\vec{u}$ in V, guarantees the existence of a linear combination which is equal to this given vector $\vec{u}$.

More formally one has

Definition 3a (Span of a set of vectors in V)

Let $Q = \{v_1, \ldots, v_k : v_i \in V\} \subset V$ be a set of vectors in V. Then

$$Sp(Q) \equiv \text{span}\, Q = \{v : v = a_1 v_1 + \cdots + a_k v_k ;\ a_i \in R\}$$

is called the "span of Q"

= "the set of linear combinations of the v_i's".

Theorem 2 $Sp(Q)$ is a subspace of V.

1. $Sp(Q) \subseteq V$
2. We have closure under addition

$$v + w = (a_1 + b_1) v_1 + \cdots + (a_k + b_k) v_k \in Sp(Q)$$

3. We have closure under scalar mplication

$$cv = ca_1 v_1 + \cdots ca_k v_k \in Sp(Q)$$

Hence $Sp(Q)$ is a subspace of V indeed.

Definition 3 b (Spanning set for V)

$Q = \{v_1, \ldots, v_k\} \subset V$ is said to be

a "spanning set for V" if $\boxed{V \subseteq Sp(Q)}$,

i.e. for any $v \in V$ $\exists$ constants $a_1, \ldots, a_k$

such that

$$v = a_1 v_1 + \cdots + a_k v_k$$

It is worthwhile to note that Def. 3b is equivalent to $V = Sp(Q)$, namely

Theorem 3

Q is a <u>spanning set</u> for $V \Leftrightarrow V = Sp(Q)$

Proof:

$\Rightarrow$:

$$\left.\begin{array}{c} \underset{\substack{\uparrow \\ \text{Def. 3b}}}{V \subseteq Sp(Q)} \\ \\ \underset{\substack{\uparrow \\ \text{Theorem 2: closure under linear combinations}}}{Sp(Q) \subseteq V} \end{array}\right\} V = Sp(Q)$$

$\Leftarrow$:

$V = Sp(Q) \Rightarrow V \subseteq sp(Q)$, i.e.

$v = a_1 v_1 + \cdots + a_k v_k$

for some $\{a_1, \ldots, a_k\}$

$\therefore Q = \{v_1, \ldots, v_k\}$ is a spanning set for V.

Example 1

Let $\mathcal{P}_n$ = set of real polynomials of degree n or less

$\{1, x, \cdots, x^n\} \equiv Q$ (= spanning set for $\mathcal{P}_n$)

$\mathcal{P}_n = Sp\{1, x, \cdots, x^n\}$

Example 2

Let

$W = \{p(x) : p(x) \in \mathcal{P}_2, \ p(0) = 0\} \subset \mathcal{P}_2$

$= \{a_2x^2 + a_1x + a_0 : p(0) = 0\} \Rightarrow$

$\Downarrow$

$a_0 = 0$

$\therefore W = \{a_2x^2 + a_1x\}$

$= Sp\{x, x^2\}$

$Q = \{x, x^2\}$ is the spanning set

Example 3 : Spanning set may be non-finite!

Let $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$ = set of all polynomials

$Q = \{1, x, \cdots, x^k, \cdots\}$; $Sp(Q) = \mathcal{P}$