LECTURE 2

Vector Spaces: Their Observational Basis

The Subspace Theorem

Spanning set

(Linear independence

For LECTURE 3

Basis; coordinates

Basis-induced isomorphism
Vector Spaces: Their Basis in Observation

Reconsider the vector space consisting of the set of inventories of a supermarket. (Nota bene: In the science of supply chain management, an inventory with a negative number of items is called a "backlog" or "backorder".) But this time consider the inventories consisting of fruits and vegetables (e.g. cucumbers, tomatoes, asparagus).

In this case one has fruit-plus-vegetable inventories. Each one is a composite inventory which consists of a fruit inventory plus a vegetable inventory. This observation is mathematized by the equation:

\[ \overrightarrow{V} = \overrightarrow{\text{fruit inventory}} + \overrightarrow{\text{vegetable inventory}} \]

\[ \equiv \overrightarrow{v_f} + \overrightarrow{v_{veg}} \]  \hspace{1cm} (1)

The set of such vectors is a new vector space, the space of fruit-plus-vegetable inventories, which is formed from the vector spaces \( V_f \) and \( V_{veg} \) and which is designated by

\[ V = \{ \overrightarrow{\text{fruit inventory}} \} \oplus \{ \overrightarrow{\text{vegetable inventory}} \} \]

\[ \equiv V_f \oplus V_{veg} \]  \hspace{1cm} (2)

In the ensuing lecture we shall identify \( V_f \) and \( V_{veg} \) as subspaces of \( V \). Furthermore, the vector space \( V \) is called the direct sum of \( V_f \) and \( V_{veg} \). It is called a direct sum, because the only inventory common to both is the trivial inventory, i.e. the zero vector:

\[ V_f \cap V_{veg} = \{ \overrightarrow{0} \} \]

Whenever that is the case, the decomposition Eq. (1) is necessarily unique. The validity of this uniqueness claim is highlighted by the question

Q: Why does this hold for every inventory in \( V \) and its answer, an observation about the nature of things:

A: Everything which exists has a specific nature:

(a) a fruit is a fruit and a vegetable is a vegetable.
(b) More generally, this is this and that is that, i.e. A is A.
(c) If it is a fruit it is not a non-fruit, i.e. it is not a vegetable; if it is a vegetable, it is not a non-vegetable, i.e. it is not a fruit.

This is Aristotle’s Law of Identity in action. It is a conceptualized observation about the nature of things. It is a pre-condition for any type of valid reasoning, inductive or deductive, in science, in engineering, in mathematics, in the humanities, indeed – in all of knowledge.

Stated negatively, this law says: Contradictions do not exist in the physical world; the existence of a contradiction is prima facia evidence for erroneous thinking.
I. The Subspace Theorem

Q: How does one determine whether a set of objects forms a vector space?

A: This task is accomplished by verifying that they obey properties 3) a-d and 4) e) of Definitions 1 on pages 1.7-1.8 in Lecture 1.

However, under certain circumstances this 8-step task can be reduced to one of two step only. The key to success lies in identifying this set of objects as a subset of some vector space, and then merely checking closure under addition and multiplication by scalars.
One starts with

Definition 2 (subspace)

Let $V, W$ both be vector spaces

$W = V$, i.e. $W$ is a subset of $V$

Then $W$ is called a subspace of $V$.

Example:

$W = \mathbb{R}^2 \subset V = \mathbb{R}^3$

So a subspace is a subset which also is a vector space.
sub(vector)spaces are easy to recognize because

**Theorem 1**

Let
1. $W \subseteq V = \text{vector space}$
2. $W$ is non-empty, i.e., $W$ is a non-empty subset of $V$

**Conclusion**

$W$ is a subspace $\iff$

1. $u + v \in W$ for all $u, v \in W$
2. $cw \in W$ for all $c \in F$

**Comment:**

More compactly, we have

$W$ is a subspace $\iff$ $cu + v \in W$

**Proof:** $\Rightarrow$ is obvious because $W$ is a vector space.

$\Leftarrow$

1. Commutativity, associativity, and distributivity are inherited from $V$.
2. $W$ contains the zero vector because $W \ni 0.0 = 0$ as shown in Lecture 1.

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1. Commutativity
2. $1 \mathbf{v} = \mathbf{v}$
3. Distributivity
4. $c(d \mathbf{v}) = (cd) \mathbf{v}$

2. W contains the zero vector (3c):
   - $W$ non-empty $\Rightarrow \exists \mathbf{v} \in W$. $W$ closed under scalar multiplication $\Rightarrow \forall \mathbf{v} \in W$. $0 \cdot \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{0} \in W$.
   - Given (shown in Lecture 1)

3. For any $\mathbf{v} \in W$, $W$ contains the additive inverse (3d):
   - Let $\mathbf{v} \in W$. Then $(1) \mathbf{v} = -\mathbf{v} \in W$

Comment:
- Theorem 1 simplifies the task of determining whether or not $W$ is a vector space: all one needs to do is verify $c \mathbf{v} + d \mathbf{w} \in W$ whenever $\mathbf{v}, \mathbf{w} \in W$. 
Example.

For $m < n$

$$P_m = \{ p(x) = a_m x^m + \ldots + a_1 x + a_0 : a_m, \ldots, a_0 \in \mathbb{R} \}$$

is a subspace of $P_n$. Indeed,

a) we know that $P_m$ is a vector space.

b) " " " " $P_m \subseteq P_n$.

Thus

c) $P_m$ is a subspace because $P_m$ is closed under addition and scalar multiplication.

II Spanning Sets

We shall now give several definitions, and from these deduce several theorems.

The difficult part of this is to come up with the definitions; this is basically an inductive process whose hard work we owe to the 18th and 19th century mathematicians.
II Spanning Sets

We shall now state several definitions and from them deduce several theorems.

A definition states the distinguishing property(ies) of a concept ["span of A"; "spanning set for V"].

The difficult part of this enterprise is to form the concept and to identify its distinguishing (i.e., essential) property. This is done by a process of inductive reasoning, which is much more difficult than deductive reasoning, which is the formulation of the theorems.
In other words, arriving at these theorems depends crucially on the concepts identified by means of their definition. The quality and validity of the theorems depends on the quality and validity of the concepts which make the theorems possible. The formation of concepts are green lights for the statements of theorems, relations between the concepts.
Comment and Outlook:

In the first lecture we considered vectors wholesale. In this lecture we shall consider them retail. In the first lecture we were interested in recognizing whether the whole set is indeed a vector space and how one does this.

In this lecture we will single out finite number of vectors and use them to coordinate the vector space. It turns out that there is an enormous amount of flexibility in establishing a coordinate system, and this flexibility is a reflection of the flexibility in performing measurements for identifying the nature of things in the physical world.
II. Spanning Set

A spanning set is a set of vectors, which, given any vector \( \mathbf{u} \) in \( V \), guarantees the existence of a linear combination which is equal to this given vector \( \mathbf{u} \).

More formally one has

**Definition 3.** (Span of a set of vectors in \( V \))

Let \( Q = \{ \mathbf{v}_1, \ldots, \mathbf{v}_k : \mathbf{v}_i \in V \} \subseteq V \) be a set of vectors in \( V \). Then

\[ \text{Sp}(Q) = \text{span} \, Q = \{ \mathbf{v} : \mathbf{v} = a_1 \mathbf{v}_1 + \cdots + a_k \mathbf{v}_k, \ a_i \in \mathbb{R} \} \]

is called the "span of \( Q \)"

= "the set of linear combinations of the \( \mathbf{v}_i \)'s."
Theorem 2. \( \text{Sp}(Q) \) is a subspace of \( V \).

1. \( \text{Sp}(Q) \subseteq V \)
2. We have closure under addition
   \[ \bar{v} + \bar{w} = (a_1 + b_1) \bar{v}_1 + \ldots + (a_n + b_n) \bar{v}_n \in \text{Sp}(Q) \]
3. We have closure under scalar multiplication
   \[ c \bar{v} = c_1 \bar{v}_1 + \ldots + c_n \bar{v}_n \in \text{Sp}(Q) \]

Hence \( \text{Sp}(Q) \) is a subspace of \( V \) indeed.

Definition 3 (Spanning set for \( V \))

\( \{ \bar{v}_1 , \ldots , \bar{v}_k \} \subseteq V \) is said to be a "spanning set for \( V \)" if \( V \subseteq \text{Sp}(Q) \), i.e., for any \( \bar{v} \in V \), \( \exists \) constants \( a_1 , \ldots , a_k \) such that

\[ \bar{v} = a_1 \bar{v}_1 + \ldots + a_k \bar{v}_k \]
It is worthwhile to note that Def. 3b
is equivalent to \( V = \text{Sp}(q) \), namely.

**Theorem 3**

\( q \) is a spanning set for \( V \iff V = \text{Sp}(q) \)

**Proof:**

\( \Rightarrow: \)

\[
V \subseteq \text{Sp}(q)
\]

\[
\begin{align*}
\text{Def. 3b} & \quad \left\{ \begin{array}{l}
V = \text{Sp}(q) \\
\text{Sp}(q) = V
\end{array} \right.
\end{align*}
\]

Theorem 2: closure under linear combinations

\( \Leftarrow: \)

\( V = \text{Sp}(q) \Rightarrow V \subseteq \text{Sp}(q) \), i.e.

\( V = a_1 v_1 + \ldots + a_k v_k \)

for some \( a_1, \ldots, a_k \)

\( \therefore q = \{ v_1, \ldots, v_k \} \) is a spanning set for \( V \).
Example 1
Let \( P_n = \) set of real polynomials of degree \( n \) or less

\[ x_1, \ldots, x_n \subseteq \mathbb{R} \] is a spanning set for \( P_n \)

\[ P_n = \text{sp} \{ x_1, \ldots, x_n \} \]

Example 2
Let

\[ W = \{ p(x) : p(x) \in P_n, p(0) = 0 \} \subset P_n \]

= \{ a_n x^n + a_1 x + a_0 ; p(0) = 0 \} \rightarrow

\[ a_0 = 0 \]

\[ W = \{ a_n x^n + a_1 x \} \]

= \text{sp} \{ x, x^2 \}

\[ Q = \{ x, x^2 \} \text{ is the spanning set} \]

Example 3: Spanning set may be non-finite!
Let \( \mathcal{P} = \bigcup_{n=0}^{\infty} P_n = \) set of all polynomials

\[ Q = \{ 1, x, x^2, x^3, \ldots \} \quad \text{sp}(Q) = \mathcal{P} \]