Vector Spaces: Their Basis in Observation

Reconsider the vector space consisting of the set of inventories of a supermarket. (*Nota bene:* In the science of supply chain management, an inventory with a negative number of items is called a "backlog" or "backorder".) But this time consider the inventories consisting of fruits <u>and</u> vegetables (e.g. cucumbers, tomatoes, asparagus).

In this case one has fruit-plus-vegetable inventories. Each one is a composite inventory which consists of a fruit inventory plus a vegetable inventory. This observation is mathematized by the equation:

$$\vec{v} = \overrightarrow{\text{fruit inventory}} + \overrightarrow{\text{vegetable inventory}} \qquad (1)$$
$$\equiv \vec{v_f} + \vec{v_{veg}} \qquad (2)$$

The set of such vectors is a new vector space, the space of fruit-plus-vegetable inventories, which is formed from the vector spaces V_f and V_{veg} and which is designated by

$$V = \{\overrightarrow{\text{fruit inventory}}\} \oplus \{\overrightarrow{\text{vegetable inventory}}\}$$
$$\equiv V_f \oplus V_{veg}$$

In the ensuing lecture we shall identify V_f and V_{veg} as *subspaces* of V. Furthermore, the vector space V is called the *direct sum* of V_f and V_{veg}. It is called a *direct* sum, because the only inventory common to both is the trivial inventory, i.e. the zero vector:

$$V_f \cap V_{veg} = \{\vec{0}\}$$

Whenever that is the case, the decomposition Eq. (1) is necessarily *unique.* The validity of this uniqueness claim is highlighted by the question

Q: Why does this hold for *every* inventory in V? and its answer, an observation about the nature of things:

A: Everything which exists has a specific nature:

  (a) a fruit is a fruit and a vegetable is a vegetable.

  (b) More generally, *this* is *this* and *that* is *that*, i.e. A is A.

  (c) If it is a fruit it is *not* a non-fruit, i.e. it is not a vegetable; if it is a vegetable, it is *not* a non-vegetable, i.e. it is not a fruit.

This is Aristotle's Law of Identity in action. It is a conceptualized observation about the nature of things. It is a pre-condition for any type of valid reasoning, inductive or deductive, in science, in engineering, in mathematics, in the humanities, indeed – in all of knowledge.

Stated negatively, this law says:
Contradictions do not exist in the physical world; the existence of a contradiction is *prima facia* evidence for erroneous thinking.