LECTURE 3

Monday

Spanning set example

Linear independence

Basis; coordinates

Basis-induced isomorphism

Example 2 (Spanning set for P_2)

a) Question: Is $Q = \{1+x, x+x^2, 1-x^2\}$ a spanning set for P_2?

Answer: Let $p = a_0 + a_1 x + a_2 x^2 \in P_2$.

The question is answered by asking whether

(*) $$(1+x)u + (x+x^2)v + (1-x^2)w = p = a_0 + a_1 x + a_2 x^2$$

can be solved for (u, v, w) for any $p \in P_2$;
i.e. do there exist (u, v, w) such that Eq. (*) is satisfied.

Equating equal powers of x one obtains

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_3 \end{bmatrix} \begin{matrix} \leftarrow x^0 \\ \leftarrow x^1 \\ \leftarrow x^2 \end{matrix}$$

To solve this system one applies Gaussian elimination to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a_0 \\ 1 & 1 & 0 & a_1 \\ 0 & 1 & -1 & a_2 \end{array}\right]$$

This is a symbolic way of writing 3 equations in the 3 unknowns u, v, w. They do not occur in this matrix because the to-be-done row operations do not alter the u, v, and w.

The Gaussian elimination yields

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a_0 \\ 0 & 1 & -1 & a_1 - a_0 \\ 0 & 0 & 0 & a_2 - a_1 + a_0 \end{array}\right]$$

Thus Q is spanning set only for those elements of P_2 for which $a_2 = a_1 - a_0$, i.e. only for polynomial whose form is

$$\{ p(x) = a_0 + a_1 x + (a_1 - a_0) x^2 \} = sp(Q) \subset V$$

Conclusion: Q is not a spanning set for V because $Sp(Q) \neq V$.

b) Question: Is $Q = \{1+x, x+x^2, 1+x^2\}$ a spanning set for V?

Answer: Yes.

Indeed, applying Gaussian elimination yields

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a_0 \\ 0 & 1 & -1 & a_1 - a_0 \\ 0 & 0 & 2 & a_2 - a_1 + a_0 \end{array}\right]$$

Reduction to echelon form yields

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}(a_0 + a_1 - a_2) \\ 0 & 1 & 0 & \frac{1}{2}(-a_0 + a_1 + a_2) \\ 0 & 0 & 1 & \frac{1}{2}(a_0 - a_1 + a_2) \end{array}\right] \begin{array}{l} \leftarrow u \\ \leftarrow v \\ \leftarrow w \end{array}$$

Thus any $p \in P_2$

$$p(x) = (1+x)u + (x+x^2)v + (1+x^2)w \quad (\star)$$

can be written as a linear combination of the elements of Q, i.e.

$$P_2 = \text{span}(Q).$$

i.e <u>Q is a spanning set for P_2.</u>

III Linear Independence

Besides the spanning property of a set of vector there is its other key property, namely its linear independence (or dependence). These concepts are identified by means of the following definitions.

Definition 4 (Linear dependence / independence)

Let V be a vector space

Let $\{v_1, \dots, v_p\} \subset V$

Consider the equation

$$a_1 v_1 + \dots + a_p v_p = \vec{0}$$

If there exists a nontrivial solution, i.e

$\exists$ $a_1, \dots, a_p$ not all zero,

then one says that $\{a_1, \dots, a_p\}$ is a linearly dependent set

Put differently

a) the set $\{v_1, \ldots, v_p\}$ is said to be linearly dependent whenever

$$a_1 v_1 + \cdots + a_p v_p = \vec{0}$$

has a non-trivial solution.

b) we have Def'n 4a

$\boxed{\begin{array}{l} a_1, \ldots, a_p \text{ not all zero} \\ \text{is a solution to} \\ a_1 v_1 + \quad + a_p v_p = \vec{0} \end{array}}$ $\iff$ $\{v_1, \ldots, v_p\}$ is a lin. dep. set

Stated still more differently, we have Def'n 4b

c) $\{v_1, \ldots, v_p\}$ is a linearly independent set if it is not linearly dependent

$\boxed{\begin{array}{l} a_1 = \cdots = a_p = 0 \text{ is the only} \\ \text{solution to} \\ a_1 v_1 + \cdots + a_p v_p = \vec{0} \end{array}}$ $\iff$ $\{v_1, \ldots, v_p\}$ is a linearly indep set.

Intermediate Summary:

So far we have formed, defined, and related the following concepts;

Vector space	Def. 1
Subspace	Def 2, Thm 1
Q = spanning set $Sp(Q)$ = span of Q	} Def. 3a
$Sp(Q)$ is a subspace	Thm 2
Spanning set for V:	Def 3b
$Sp(Q) = V$	Thm 3
Linearly dependent set	Def. 4a
Linearly independent set	Def. 4b

IV Basis For a Vectorspace

By applying the concept of linear independence to a spanning set, say Q, one arrives at the concept of a basis for $Sp(Q)$.

Example 2

Again, consider $Q = \{1+x, x+x^2, 1-x^2\}$

and Eq. (*) on page 3.1 with $a_0 = a_1 = a_2 = 0$,

$$(1+x)u + (x+x^2)v + (1-x^2)w = \vec{0} \text{ (= zero polynomial)}$$

The result of the reduction to echelon form yields

$$u = 0$$
$$v = 0$$
$$w = 0$$

and this is the only solution. Consequently, Q is a spanning set (for P_2) which is linearly independent.

A linearly independent spanning set, such as Q, is called a basis for P_2.

Thus we have the following

Definition 5a (Basis for a vector space)

(i) Let V = vector space

(ii) Let $B = \{v_1, \ldots, v_p\} \subset V$ be a spanning set for V, i.e.

(iii) $V = Sp(B)$ ("B is a spanning set for V")

If B is linearly independent then B is called a basis. In other words, a linearly independent spanning set is a basis (for the spanned space).