LECTURE 3 Monday

Spanning set example
Linear independence
Basis, coordinates
Basis-induced isomorphism
Example 1 (Spanning set for \( \mathbb{P}_2 \))

a) Question: Is \( Q = \{1+x, x+x^2, 1-x^2\} \) a spanning set for \( \mathbb{P}_2 \)?

Answer: Let \( p = a_0 + a_1 x + a_2 x^2 \in \mathbb{P}_2 \).

The question is answered by asking whether

\[ (1+x)u + (x+x^2)v + (1-x^2)w = p = a_0 + a_1 x + a_2 x^2 \]

can be solved for \((u, v, w)\) for any \( p \in \mathbb{P}_2 \);

i.e., do there exist \((u, v, w)\) such that Eq. (1) is satisfied.

Equating equal powers of \( x \) one obtains

\[
\begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
=
\begin{bmatrix}
a_0 \\
a_1 \\
a_2
\end{bmatrix}
\]

To solve this system one applies Gaussian elimination to the augmented matrix

\[
\begin{bmatrix}
1 & 0 & 1 & a_0 \\
1 & 1 & 0 & a_1 \\
0 & 1 & -1 & a_2
\end{bmatrix}
\]
This is a symbolic way of writing 3 equations in the 3 unknowns \( u, v, w \). They do not occur in this matrix because the to-be-done row operations do not alter the \( u, v, \) and \( w \).

The Gaussian elimination yields

\[
\begin{bmatrix}
1 & 0 & 1 & \mid & a_0 \\
0 & 1 & -1 & \mid & a_1 - a_0 \\
0 & 0 & 0 & \mid & a_2 - a_1 + a_0 \\
\end{bmatrix}
\]

Thus \( Q \) is spanning set only for those elements of \( R^n \) for which

\[ a_2 = a_1 - a_0, \]

i.e., only for polynomial whose form is

\[ \{ p(x) = a_0 + a_1 x + (a_1 - a_0) x^2 \} = \text{sp}(Q) \subset \mathbb{V} \]

Conclusion: \( Q \) is not a spanning set for \( \mathbb{V} \) because \( \text{sp}(Q) \neq \mathbb{V} \).
b) Question: Is \( Q = \{1+x, x+x^2, 1+x^2\} \) a spanning set for \( V \)?

Answer: Yes.

Indeed, applying Gaussian elimination yields

\[
\begin{bmatrix}
1 & 0 & 1 & | & a_0 \\
0 & 1 & -1 & | & a_1 - a_0 \\
0 & 0 & 2 & | & a_2 - a_1 + a_0
\end{bmatrix}
\]

Reduction to echelon form yields

\[
\begin{bmatrix}
1 & 0 & 0 & | & \frac{1}{2} (b_0 + a_1 - a_2) \\
0 & 1 & 0 & | & \frac{1}{2} (-a_0 + a_1 + a_2) \\
0 & 0 & 1 & | & \frac{1}{2} (a_0 - a_1 + a_2)
\end{bmatrix}
\]

Thus any \( p \in \mathbb{P}_2 \)

\[ p(x) = (1+x)u + (x+x^2)v + (1+x^2)w \] (\( \star \))

can be written as a linear combination of the elements of \( Q \), i.e.

\[ \mathbb{P}_2 = \text{span}(Q) \]

i.e \( Q \) is a spanning set for \( \mathbb{P}_2 \).
III Linear Independence

Besides the spanning property of a set of vectors, there is its other key property, namely its linear independence (or dependence). These concepts are identified by means of the following definitions.

Definition 4 (Linear dependence/independence)

Let \( V \) be a vector space.

Let \( \mathbf{v}_1, \ldots, \mathbf{v}_p \in V \)

Consider the equation

\[ 0, \mathbf{v}_1 + \cdots + a_p \mathbf{v}_p = \mathbf{0} \]

If there exists a nontrivial solution \( i.e. \)

\[ \equiv a_1, \ldots, a_p \text{ not all zero} \]

then one says that \( \mathbf{v}_1, \ldots, \mathbf{v}_p \) is a linearly dependent set
Put differently

(a) The set \( \{v_1, \ldots, v_p\} \) is said to be linearly dependent whenever
\[
0 = a_1 v_1 + \cdots + a_p v_p
\]
has a non-trivial solution.

(b) We have Definition 4a
\[
a_1, \ldots, a_p \text{ not all zero}
\]

is a solution to
\[
0 = a_1 v_1 + \cdots + a_p v_p
\]
\( \iff \) \( \{v_1, \ldots, v_p\} \) is a linearly dependent set

Stated still more differently, we have Definition 4b

(c) \( \{v_1, \ldots, v_p\} \) is a linearly independent set if it is not linearly dependent

\[
a_1 = \cdots = a_p = 0 \text{ is the only solution to}
\]
\[
0 = a_1 v_1 + \cdots + a_p v_p
\]
\( \iff \) \( \{v_1, \ldots, v_p\} \) is a linearly independent set.
Intermediate Summary:

So far we have formed, defined, and related the following concepts:

- Vector space: Def. 1
- Subspace: Def 2, Thm 1

\( \mathcal{A} = \text{spanning set} \)

\( \text{Sp}(\mathcal{A}) = \text{span of } \mathcal{A} \) \( \text{Def. 3a} \)

\( \text{Sp}(\mathcal{A}) \) is a subspace \( \text{Thm 2} \)

Spanning set for \( V \) \( \text{Def 3b} \)

\( \text{Sp}(\mathcal{A}) = V \) \( \text{Thm 3} \)

- Linearly dependent set: Def. 49
- Linearly independent set: Def. 48
IV Basis For a Vector space

By applying the concept of linear independence to a spanning set, say \( \delta \), one arrives at the concept of a basis for \( \text{Sp}(\delta) \).

Example 2

Again, consider \( \delta = \{1+x, x+x^2, 1-x^2\} \)

and Eq. (*) on page 3.1 with \( a_0 = 0, a_1 = 0, a_2 = 0 \),

\((1+x)u + (x+x^2)v + (1-x^2)w = \vec{0} \) \( (= \text{zero polynomial}) \)

The result of the reduction to echelon form yields

\[ u = 0 \]
\[ v = 0 \]
\[ w = 0 \]

and this is the only solution. Consequently, \( \delta \) is a spanning set \( (\text{for } \mathbb{R}^3) \) which is linearly independent.
A linearly independent spanning set such as $\{e_1\}$ is called a basis for $\mathbb{R}^2$.
Thus we have the following:

Definition 5a (Basis for a vector space)

1. Let $V = \text{vector space}$

2. Let $B = \{v_1, \ldots, v_p\} \subset V$ be a spanning set for $V$, i.e.

   $V = \text{Sp}(B)$ ("$B$ is a spanning set for $V$")

If $B$ is linearly independent then $B$ is called a basis. In other words, a linearly independent spanning set is a basis (for the spanned space).