LECTURE 31

I) The Four Fundamental Subspaces of a Matrix.

II) Row Rank = Column Rank

III) Pairwise Orthogonality

IV) Direct Sum Decomposition of Domain and Target space

V Orthogonal Complementarity.

I) THE FOUR FUNDAMENTAL SUBSPACES OF A LINEAR TRANSFORMATION.

Consider an $m \times n$ matrix

$$A : R^n \to R^m \quad ; \quad A = \underbrace{\begin{bmatrix} \quad & \quad & \quad \\ & & \end{bmatrix}}_{n} \Big\} m$$

and its transpose

$$A^T : R^m \to R^n$$

Thus one may have

$$A = \underbrace{\begin{bmatrix} \; \\ \; \\ \; \end{bmatrix}}_{n} \Bigg\} m \quad \text{and } A^T = \underbrace{\begin{bmatrix} \quad & \quad & \quad \end{bmatrix}}_{m} \Big\} n$$

"skinny" "squat"

or

$$A = \underbrace{\begin{bmatrix} \quad & \quad & \quad \end{bmatrix}}_{n} \Big\} m \quad \text{and } A^T = \underbrace{\begin{bmatrix} \; \\ \; \\ \; \end{bmatrix}}_{m} \Bigg\} n$$

"squat" "skinny"

Such matrices are characterized by its four fundamental subspaces;

$\mathcal{R}(A) = (\text{column space of } A) \subset R^m$; $\dim = r$

$\mathcal{N}(A) = (\text{null space of } A) \subset R^n$; $\dim = n - r$

$\mathcal{R}(A^T) = (\text{row space of } A) \subset R^n$; $\dim = r$

$\mathcal{N}(A^T) = (\text{left nullspace of } A) \subset R^m$; $\dim = m - r$

II) ROW RANK = COLUMN RANK.

It is blindingly obvious that row space $\mathcal{R}(A^T)$ and the columnspace $\mathcal{R}(A)$ of A are in general two entirely different vector spaces. Nevertheless one has the following

<u>Theorem</u> (Row rank = column rank)

$$\dim \mathcal{R}(A) = \dim \mathcal{R}(A^T)$$

i.e. column rank of A = row rank of A.

The validity of this theorem is established by means of a 4 Step line of reasoning;

Step 1 Row reduction to echelon form

$$A=\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \xrightarrow[\text{reduction}]{\text{row}} U=\left.\begin{bmatrix} [\text{---} & u_1 & \text{---}] \\ 0 & & \\ 0\;0 & & \\ 0\;0\cdots & [\text{---}\, u_k & \text{---}] \\ \text{---} & 0 & \text{---} \end{bmatrix}\right\} \begin{matrix} k \\ m \end{matrix}$$

Step 2

Noting that $\{u_1, \ldots, u_k\}$ is a basis for $\mathbb{R}(A^T)$, we have

$$\boxed{\dim \mathbb{R}(A^T) = k} \qquad (= \text{"row rank"})$$

Step 3

Note that:

(i) The reversibility of every row operation guarantees that every row of A is a linear combination of the the rows $u_1, \ldots, u_k$. The coefficients of these combinations form a matrix $[c_{ij}]$ which relates A to U:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1k} & \cdots 0 \cdots \\ c_{21} & \cdots & c_{2k} & \cdots 0 \cdots \\ \vdots & & \vdots & \\ c_{m1} & \cdots & c_{mk} & \cdots 0 \cdots \end{bmatrix} \begin{bmatrix} [\text{---} & u_1 & \text{---}] \\ 0 & & \\ 0\;0 & & \\ 0\;0\cdots 0 & [\text{---}\, u_k & \text{---}] \\ \text{---} & 0 & \text{---} \end{bmatrix}$$

$$A \quad = \quad C \; U$$

The matric C is a unique expression of the reversible row reduction process $A \leftrightarrow U$

Indeed, we have

$$i^{th} \text{ row of } A = \sum_{\ell=1}^{k} c_{i\ell}\, [\text{---}\, u_\ell \,\text{---}]$$

Step 4

On the other hand, CU is a family of linear combinations of the columns of C, each linear combination being a column of A:

$$j^{th} \text{ column of } A = \sum_{\ell=1}^{k} \begin{bmatrix} | \\ c_\ell \\ | \end{bmatrix} u_{\ell j}$$

Thus

$$\mathbb{R}(A) = sp\{c_1, \ldots, c_k\},$$

and so

$$\dim \mathbb{R}(A) \leq k = \dim \mathbb{R}(A^T)$$

By reversing the roles of A and A^T one obtains

$$\dim \mathcal{R}(A^T) \le \dim \mathcal{R}(A).$$

and hence

$$\boxed{\dim \mathcal{R}(A^T) = \dim \mathcal{R}(A)} = \text{"rank of } A\text{"}$$

Thus, although $\mathcal{R}(A^T)$ and $\mathcal{R}(A)$ are different vector spaces, their dimensions are the same.

III) PAIRWISE ORTHOGONALITY

The second important ppty of the Four Fundamental subspaces is their pairwise orthogonality,

In R^m $\quad \mathcal{R}(A) \perp \mathcal{N}(A^T)$

In R^n $\quad \mathcal{R}(A^T) \perp \mathcal{N}(A)$

This property is based on the

Definition

Two subspaces V and W of R^n are said to be orthogonal ($V \perp W$) if every $v \in V$ is orthogonal to every $w \in W$,

and on the

Theorem

For any $m \times n$ matrix $A: R^n \to R^m$ on has

(a) $\mathcal{N}(A) \perp \mathcal{R}(A^T)$ in R^n

(b) $\mathcal{N}(A^T) \perp \mathcal{R}(A)$ in R^m.

The proof is by inspecting the elements of the two nullspaces.

(a) $w \in \mathcal{N}(A) \Rightarrow Aw = 0$

$$\begin{bmatrix} \text{row } 1 \\ \vdots \\ \text{row } m \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \qquad m \text{ equations}$$

$$(i^{th} \text{ column of } A^T)^T w = 0$$

$$\mathcal{R}(A^T) \perp w = 0 \qquad \forall w \in \mathcal{N}(A)$$

$$\therefore \mathcal{R}(A^T) \perp \mathcal{N}(A).$$

(b) $y \in \mathcal{N}(A^T) \Rightarrow \left(y^T A = 0 \Longleftrightarrow A^T y = 0 \right)$

$$[y_1 \cdots y_m] \begin{bmatrix} | & & | \\ \text{col } 1 & \cdots & \text{col } n \\ | & & | \end{bmatrix} = [0 \cdots 0] \qquad n \text{ eq'ns}$$

$$y \perp \mathcal{R}(A) \qquad \forall y \in \mathcal{N}(A^T)$$

$$\therefore \mathcal{N}(A^T) \perp \mathcal{R}(A)$$

Comment: These orthogonalities imply that A induces a unique decomposition of the domain space R^n and of the target space R^m as follows

a) domain $A = R^n = \mathcal{N}(A) \oplus \mathcal{R}(A^T)$

b) domain A^T = target $A = R^m = \mathcal{N}(A^T) \oplus \mathcal{R}(A)$.

Comment:

The orthogonalies of the subspaces of A constitutes the Fundamental Theorem of Linear Algebra.

It is stated ~~most~~ succinctly in terms of orthogonal complements.

Definition

Given a subspace $W \subset R^n$, the space of vectors orthogonal to W is called the orthogonal complement of W and is denoted by $W_\perp$.

FUNDAMENTAL THEOREM OF LINEAR ALGEBRA

$$\mathcal{N}(A) = \left(\mathcal{R}(A^T)\right)_\perp \qquad \mathcal{R}(A^T) = \left(\mathcal{N}(A)\right)_\perp$$

$$\mathcal{N}(A^T) = \left(\mathcal{R}(A)\right)_\perp \qquad \mathcal{R}(A) = \left(\mathcal{N}(A^T)\right)_\perp$$