LECTURE 31

I) The Four Fundamental Subspaces of a Matrix

II) Row Rank = Column Rank

III) Pairwise Orthogonality of A

IV) Direct Sum Decomposition of Domain and Target space

V Orthogonal Complementarity

Discussion Topics

I) THE FOUR FUNDAMENTAL SUBSPACES OF A LINEAR TRANSFORMATION.

Consider an $m \times n$ matrix

$A : R^n \to R^m$; $A = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix}}_{n \text{ "squat"}} \Big\} m$ or $A = \underbrace{\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}}_{n \text{ "skinny"}} \Big\} m$

and its transpose

$A^T : R^m \to R^n$; $A^T = \underbrace{\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}}_{m \text{ "skinny"}} \Big\} n$ or $A^T = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix}}_{m \text{ "squat"}} \Big\} n$

Thus one may have

$A = \underbrace{\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}}_{n} \Big\} m$ and $A^T = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix}}_{m} \Big\} n$

"skinny" "squat"

or

$A = \underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \end{bmatrix}}_{n} \Big\} m$ and $A^T = \underbrace{\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}}_{m} \Big\} n$

"squat" "skinny"

Such matrices are characterized by its four fundamental subspaces;

$\mathcal{R}(A) = (\text{column space of } A) \subset R^m$; dim $= r$

$\mathcal{N}(A) = (\text{null space of } A) \subset R^n$; dim $= n - r$

$\mathcal{R}(A^T) = (\text{row space of } A) \subset R^n$; dim $= r$ (←?)

$\mathcal{N}(A^T) = (\text{left nullspace of } A) \subset R^m$; dim $= m - r$ (←?)

II) ROW RANK = COLUMN RANK.

It is blindingly obvious that row space $\mathcal{R}(A^T)$ and the columnspace $\mathcal{R}(A)$ of A are in general two entirely different vector spaces. Nevertheless one has the following

Theorem (Row rank = column rank)

$$\dim \mathcal{R}(A) = \dim \mathcal{R}(A^T)$$

i.e. column rank of A = row rank of A.

THEOREM: Row rank=Column rank

The validity of this theorem is established by means of a 4 step process.
STEP 1
Row reduction to echelon form.

$$A=\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}. \rightarrow U=\begin{bmatrix} d_1 & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ 0 & \cdots & 0 & d_2 & * & * & * & * & * & * & * & * & * & * & * \\ 0 & \cdots & 0 & 0 & \cdots & 0 & d_3 & * & * & * & * & * & * & * & * \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & & & & & & & & & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & d_k & * & * & * & * & * & * \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & & & & & & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \tag{1}$$

The matrix A is $m \times n$. The matrix U, which is also $m \times n$, has k non-zero rows with non-zero pivot $d_1, d_2, d_3, \cdots, d_k$ and $(m-k)$ rows which are zero.
STEP 2
The row space of A and of U are the same. Row operations guarantee this. Furthermore, being in reduced form, U has a linearly independent set of k

row vectors $u_1, u_2, u_3, \cdots, u_k$:

$$U = \begin{bmatrix} [& \text{———} & & & & u_1 & & \text{———} & &] \\ 0 & \cdots & 0 & [& \text{———} & & u_2 & & \text{———} &] \\ 0 & \cdots & 0 & 0 & \cdots & 0 & [& \text{———} \; u_3 \; \text{———} & &] \\ \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & [\text{——} \; u_k \; \text{——}] \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \; \cdots \; 0 \; \cdots \; 0 \\ \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \; \cdots \; 0 \; \cdots \; 0 \end{bmatrix}$$

$$= \begin{bmatrix} \text{———} \; \vec{u}_1 \; \text{———} \\ \text{———} \; \vec{u}_2 \; \text{———} \\ \text{———} \; \vec{u}_3 \; \text{———} \\ \vdots \\ \text{———} \; \vec{u}_k \; \text{———} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

These row vectors form a basis for the row space of U and for that of A. Consequently,

$$\text{“row rank”} \equiv \dim \mathcal{R}(A^T) = k \tag{2}$$

STEP 3

Row operations are reversible. Consequently, the row reduction process is reversible. Thus the rows of A are linear combinations of the rows of U, namely, the k basis elememts $\vec{u}_1, \vec{u}_2, \vec{u}_3, \cdots$, and $\vec{u}_k$:

$$[a_{11} \cdots a_{1n}] = c_{11}\vec{u}_1 + c_{12}\vec{u}_2 + \cdots c_{1k}\vec{u}_k$$

$$\vdots$$

$$[a_{m1} \cdots a_{mn}] = c_{m1}\vec{u}_1 + c_{m2}\vec{u}_2 + \cdots c_{mk}\vec{u}_k$$

or

$$
\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{m \times n} = \underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1k} & 0 & \cdots & 0 \\ c_{21} & c_{22} & \cdots & c_{2k} & 0 & \cdots & 0 \\ \vdots & & & & & & \\ c_{m1} & c_{m2} & \cdots & c_{mk} & 0 & \cdots & 0 \end{bmatrix}}_{m \times m} = \underbrace{\begin{bmatrix} \text{———} \ \vec{u}_1 \ \text{———} \\ \text{———} \ \vec{u}_2 \ \text{———} \\ \vdots \\ \text{———} \ \vec{u}_k \ \text{———} \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{m \text{ rows} \times n \text{ columns}}
$$

or more compactly,

$$A = CU.$$

The coefficients c_{ij}, which make up the $m \times m$ matrix C, are uniquely determined by the row reduction process $A \to U$.

STEP 4

This step contrasts two different ways of viewing the product of two matrices. Instead of viewing $A = CU$ as reconstructing the rows of A as linear combinations of the rows $u_1, u_2, \cdots, u_k$ of U, one now views the *columns* of A as linear combinations of the *columns of* C. Note that there are only k non-zero such columns. Consequently,

$$\mathcal{R}(A) = Span\left(\left\{\begin{bmatrix} c_{1i} \\ \vdots \\ c_{mi} \end{bmatrix}\right\}_{i=1}^{k}\right)$$

and therefore

$$\dim \mathcal{R}(A) \leq k = \dim \mathcal{R}(A^T) \quad \text{(using Eq.(2) on page 2).}$$

Finally, by reversing the roles of A and A^T and applying the previous four steps one obtainsu dsainalogous iequality

$$\dim \mathcal{R}(A^T) \leq \dim \mathcal{R}(A).$$

These last two inequalities imply

$$\dim \mathcal{R}(A^T) = \dim \mathcal{R}(A),$$

i.e. for any matrix A, its row rank equals its column rank indeed.

(see previous type set pages)

The validity of this theorem is established by means of a 4 step process

Step 1 Row reduction to echelon form

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{mn} & \cdots & a_{mn} \end{bmatrix} \xrightarrow[\text{reduction}]{\text{row}} U = \begin{bmatrix} d_1 * * * * * \\ 0 \cdots 0 \; d_2 * * \\ 0 \cdots 0 \; d_3 * * \\ \vdots \\ 0 \cdots 0 \cdots 0 \; 0 \cdots 0 \; d_k * * * * \\ 0 \quad\quad 0 \cdots 0 \end{bmatrix}$$

$m \times n$

m

with k non-zero rows and $m-k$ row which are zero

Step 2

Note that, being in echelon form, U has k linearly independent rows. Moreover, they span the row space of U. Thus $\{u_1, u_2, \ldots, u_k\}$ is a basis for the row space of U.

The row reduction process consists of
(a) interchanging a pair of rows
(b) multiplying a row by a non-zero constant
(c) add one row to another.

(see previous typeset pages)

STEP 2

The row space of A and of U are the same. Row operation guarantee this.

Furthermore, being in echelon form, U has k linearly independent rows:

$$U = \begin{bmatrix} -\vec{u}_1- \\ -\vec{u}_2- \\ -\vec{u}_3- \\ \vdots \\ -\vec{u}_k- \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} [\text{——} u_1 \text{——}] \\ 0\cdots0[\text{——} u_2 \text{——}] \\ 0\cdots0\;0\cdots0[\text{——} u_3 \text{——}] \\ \vdots \\ 0\cdots0\;0\cdots0\;0\cdots0[-u_k-] \\ 0\cdots0\;0\cdots0\;0\cdots0\;\cdots\;0 \\ 0 \quad \vdots \quad 0 \\ 0\cdots0\;\;0\;0\;\;0\cdots0 \end{bmatrix}$$

These row vectors form a basis for the row space of U and for that of A. Consequently

$$\text{"row rank"} \equiv \dim R(A^T) = k$$

(See previous typeset pages)

STEP 3

Row operations are reversible.

Consequently, the row reduction process is reversible.

Thus the rows of A are linear combinations of the rows of U, namely the k basis elements $\vec{u}_1, \vec{u}_2, \ldots,$ and $\vec{u}_k$:

$$[a_{11} \cdots a_{1n}] = c_{11}\vec{u}_1 + c_{12}\vec{u}_2 + \cdots + c_{1k}\vec{u}_k$$

$$\vdots$$

$$[a_{m1} \cdots a_{mn}] = c_{m1}\vec{u}_1 + c_{m2}\vec{u}_2 + \cdots + c_{mk}\vec{u}_k$$

or

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{m1} & & a_{mn} \end{bmatrix} = \underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1k} & 0 & \cdots & 0 \\ c_{21} & c_{22} & \cdots & c_{2k} & 0 & \cdots & 0 \\ & & & & \vdots & & \\ c_{m1} & c_{m2} & \cdots & c_{mk} & 0 & \cdots & 0 \end{bmatrix}}_{m \times m} \underbrace{\begin{bmatrix} -u_1- \\ -u_2- \\ \vdots \\ -u_k- \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{n \text{ column}} \left.\vphantom{\begin{bmatrix} 1\\1\\1\\1\\1\\1\\1 \end{bmatrix}}\right\} m \text{ rows}$$

or more compactly

$$A = CU,$$

(See previous typeset pages)

The constants c_{ij}, which make up the $m \times m$ matrix C, are uniquely determined by the row reduction process $A \to U$.

STEP 4

This step is the most subtle:

Instead of viewing CU as reconstructing the rows of A in terms of the rows $u_1, \dots, u_k$ of U, one now views CU as reconstructing the columns of A in terms of the columns of C.

$\boxed{\text{Note that there are only } k \text{ of them.}}$

Consequently

$$\mathcal{R}(A) = \text{span}\left\{ \begin{bmatrix} c_{1i} \\ \vdots \\ c_{mi} \end{bmatrix} \right\}_{i=1}^{k}$$

(see previous typeset pages)

and

$$\dim R(A) \le k = \dim R(A^T) \quad (*)$$

Now reverse the roles of A and A^T. The result is

$$\dim R(A^T) \le \dim R(A) \quad (**)$$

It follows that

$$\dim R(A^T) = \dim R(A)$$

i.e

$$\text{row rank} = \text{column rank}.$$

III) PAIRWISE ORTHOGONALITY

The second important ppty of 31.6

the Four Fundamental subspaces is their

pairwise orthogonality,

$$\left.\begin{array}{ll} \text{In } R^m & \mathcal{R}(A) \perp \mathcal{N}(A^T) \\ \text{In } R^n & \mathcal{R}(A^T) \perp \mathcal{N}(A) \end{array}\right\} A: R^n \to R^m \text{ ; } A \text{ is } m \times n$$

This property is based on the

<u>Definition</u>

Two subspaces V and W of R^n are said to be <u>orthogonal</u> ($V \perp W$) if every $v \in V$ is orthogonal to every $w \in W$,

and on the

<u>Theorem</u>

For any $m \times n$ matrix $A: R^n \to R^m$ on has

(a) $\mathcal{N}(A) \perp \mathcal{R}(A^T)$ in R^n

(b) $\mathcal{N}(A^T) \perp \mathcal{R}(A)$ in R^m

The proof is by inspecting the elements of the two nullspaces. 31.7

(a) $w \in \mathcal{N}(A) \Rightarrow Aw = 0$

$$\begin{bmatrix} \text{row } 1 \\ \vdots \\ \text{row } m \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \qquad m \text{ equations}$$

$$(i^{th} \text{ row of } A)\, w = (i^{th} \text{ column of } A^T)^T w = 0 \qquad i = 1, 2, \ldots m$$

$$\mathcal{R}(A^T) \perp w = 0 \qquad \forall\, w \in \mathcal{N}(A)$$

$$\therefore\ \mathcal{R}(A^T) \perp \mathcal{N}(A).$$

(b) $y \in \mathcal{N}(A^T) \Rightarrow \left(y^T A = 0 \Leftrightarrow \underset{y \in \mathcal{N}(A^T)}{A^T y = 0} \right)$

$$[y_1 \cdots y_m] \begin{bmatrix} | & & | \\ \text{col } 1 & \cdots & \text{col } n \\ | & & | \end{bmatrix} = [0 \cdots 0] \qquad n \text{ eq'ns.}$$

$$y \perp \mathcal{R}(A) \qquad \forall\, y \in \mathcal{N}(A^T)$$

$$\therefore\ \mathcal{N}(A^T) \perp \mathcal{R}(A)$$

Comment: These orthogonalities imply that A induces a unique decomposition of the domain space R^n and of the target space R^m as follows

$$\boxed{\begin{aligned} &a)\ \text{domain } A = R^n = \mathcal{N}(A) \oplus R(A^T) \\ &b)\ \text{domain } A^T = \\ &\quad = \text{target } A = R^m = \mathcal{N}(A^T) \oplus \mathcal{R}(A). \end{aligned}}$$

where $\mathcal{N}(A) \cap \mathcal{R}(A^T) = \{0\}$ in R^n

$$\mathcal{N}(A^T) \cap \mathcal{R}(A) = \{0\} \text{ in } R^m$$

Thus $w \in \mathcal{N}(T) \cap \underline{R(A^T)}$

$$w: \quad \begin{bmatrix} row(1) \\ \vdots \\ row(m) \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = 0 \qquad w^T = \sum_{k=1}^{m} \alpha_k \, row(k)$$

$$\left.\begin{aligned} &w^T w = \sum \alpha_k \, row(k) \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = 0 \\ &\text{but } w^T w = (w_1)^2 + \dots + (w_n)^2 \end{aligned}\right\} \Rightarrow w = 0$$

Comment:
The orthogonalies of the subspaces of A constitutes the Fundamental Theorem of Linear Algebra.
It is stated most succinctly in terms of orthogonal complements.

Definition
Given a subspace $W \subset R^n$, the space of vectors orthogonal to W is called the orthogonal complement of W and is denoted by $W_\perp$.

FUNDAMENTAL THEOREM OF LINEAR ALGEBRA

$$\mathcal{N}(A) = \left(\mathcal{R}(A^T)\right)_\perp \qquad \mathcal{R}(A^T) = \left(\mathcal{N}(A)\right)_\perp$$

$$\mathcal{N}(A^T) = \left(\mathcal{R}(A)\right)_\perp \qquad \mathcal{R}(A) = \left(\mathcal{N}(A^T)\right)_\perp$$