Row rank=Column rank

The validity of this theorem is established by means of a 4 step process.
STEP 1
Row reduction to echelon form.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \to U = \begin{bmatrix} d_1 & * & * & * & * & * & * & * & * & * & * & * & * & * & * \\ 0 & \cdots & 0 & d_2 & * & * & * & * & * & * & * & * & * & * & * \\ 0 & \cdots & 0 & 0 & \cdots & 0 & d_3 & * & * & * & * & * & * & * & * \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & & & & & & & & & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & d_k & * & * & * & * & * & * \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & & & & & & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \tag{1}$$

The matrix A is $m \times n$. The matrix U, which is also $m \times n$, has k non-zero rows with non-zero pivot $d_1, d_2, d_3, \cdots, d_k$ and $(m-k)$ rows which are zero.
STEP 2
The row space of A and of U are the same. Row operations guarantee this. Furthermore, being in reduced form, U has a linearly independent set of k

row vectors $u_1, u_2, u_3, \cdots, u_k$:

$$U = \begin{bmatrix} [\text{———————} & & u_1 & & \text{———————}] \\ 0 \cdots 0 & [\text{—————} & u_2 & \text{—————}] & \\ 0 \cdots 0 & 0 \cdots 0 & [\text{———} & u_3 & \text{———}] \\ \vdots & & & & \\ 0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 & [\text{——} \; u_k \; \text{——}] & \\ 0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 & \cdots 0 \\ \vdots & & & & \\ 0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 & 0 \cdots 0 & \cdots 0 \end{bmatrix}$$

$$= \begin{bmatrix} \text{——} \; \vec{u}_1 \; \text{——} \\ \text{——} \; \vec{u}_2 \; \text{——} \\ \text{——} \; \vec{u}_3 \; \text{——} \\ \vdots \\ \text{——} \; \vec{u}_k \; \text{——} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

These row vectors form a basis for the row space of U and for that of A. Consequently,

$$\text{"row rank"} \equiv \dim \mathcal{R}(A^T) = k \tag{2}$$

STEP 3

Row operations are reversible. Consequently, the row reduction process is reversible. Thus the rows of A are linear combinations of the rows of U, namely, the k basis elememts $\vec{u}_1, \vec{u}_2, \vec{u}_3, \cdots$, and $\vec{u}_k$:

$$\begin{aligned} [a_{11} \cdots a_{1n}] &= c_{11}\vec{u}_1 + c_{12}\vec{u}_2 + \cdots c_{1k}\vec{u}_k \\ &\vdots \\ [a_{m1} \cdots a_{mn}] &= c_{m1}\vec{u}_1 + c_{m2}\vec{u}_2 + \cdots c_{mk}\vec{u}_k \end{aligned}$$

or

$$\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{m \times n} = \underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1k} & 0 & \cdots & 0 \\ c_{21} & c_{22} & \cdots & c_{2k} & 0 & \cdots & 0 \\ \vdots & & & & & & \\ c_{m1} & c_{m2} & \cdots & c_{mk} & 0 & \cdots & 0 \end{bmatrix}}_{m \times m} = \underbrace{\begin{bmatrix} \text{------} & \vec{u}_1 & \text{------} \\ \text{------} & \vec{u}_2 & \text{------} \\ & \vdots & \\ \text{------} & \vec{u}_k & \text{------} \\ & 0 & \\ & \vdots & \\ & 0 & \end{bmatrix}}_{m \text{ rows} \times n \text{ columns}}$$

or more compactly,

$$A = CU.$$

The coefficients c_{ij}, which make up the $m \times m$ matrix C, are uniquely determined by the row reduction process $A \to U$.

STEP 4

This step contrasts two different ways of viewing the product of two matrices. Instead of viewing $A = CU$ as reconstructing the rows of A as linear combinations of the rows $u_1, u_2, \cdots, u_k$ of U, one now views the *columns* of A as linear combinations of the *columns of* C. Note that there are only k non-zero such columns. Consequently,

$$\mathcal{R}(A) = Span\left(\left\{\begin{bmatrix} c_{1i} \\ \vdots \\ c_{mi} \end{bmatrix}\right\}_{i=1}^{k}\right)$$

and therefore

$$\dim \mathcal{R}(A) \le k = \dim \mathcal{R}(A^T) \quad \text{(using Eq.(2) on page 2).}$$

Finally, by reversing the roles of A and A^T and applying the previous four steps one obtains the analogous iequality

$$\dim \mathcal{R}(A^T) \le \dim \mathcal{R}(A).$$

These last two inequalities imply

$$\dim \mathcal{R}(A^T) = \dim \mathcal{R}(A),$$

i.e. for any matrix A, its row rank equals its column rank indeed.