LECTURE 32

Time Invariant ("Autonomous") Linear Systems.

I. Solution via Taylor Series

II. Solution via Eigenvalues & Eigenvectors

A) Algebraic Analysis

B) Comments

C) Geometric Analysis

D) Recommendation.

III. Diagonal Form of a Matrix } next Lecture

TIME INVARIANT (AUTONOMOUS)

LINEAR SYSTEMS

Consider the system of differential equations with constant coefficients

$$\frac{d\vec{u}(t)}{dt} = A\vec{u}(t) \qquad A: V=R^n \rightarrow V=R^n$$

Find $\vec{u}(t)$ subject to the initial condition $\vec{u}(0)=\vec{u}_0$

Comment:
Here we develop the mathematical theory for a linear system represented by a constant $n\times n$ matrix A.

There are two mathematical ways of solving this dynamical system problem, namely via

I. Taylor series method

II. Eigenvector method

I. Taylor Series Method

This method can be used for any matrix A as long as it is a constant matrix. Indeed one has

$$\frac{du(t)}{dt} = Au(t)$$

$$\frac{d^2u(t)}{dt^2} = A\frac{du}{dt} = A^2u(t)$$

$$\frac{d^3u}{dt^3} = A^3u$$

The Taylor series expansion around $t=0$ yields

$$\vec{u}(t) = u(0) + \left.\frac{du}{dt}\right|_0 t + \left.\frac{d^2u}{dt^2}\right|_0 \frac{t^2}{2!} + \cdots$$

$$= \underbrace{\left[I + At + \frac{1}{2!}A^2t^2 + \cdots\right]}_{\equiv\, e^{At}} \vec{u}(0)$$

$\equiv e^{At}$ whenever the series converges.

Thus one has

$$\boxed{\vec{u}(t) = e^{At}\vec{u}_0}$$

and

$$\vec{u}(t_1+t_2) = e^{At_1}e^{At_2}\vec{u}_0 = e^{A(t_1+t_2)}\vec{u}_0$$

One says that e^{At} is the time xlation operator of the system.
Aside from the fact that A is said to be the generator of time translations of the system, there is not too much one can say about it without examining the structure of the $n \times n$ matrix A.

This deficiency is remedied by the

II. Eigenvector Method

The virtue of this method is that it finds and uses the eigenvalues and eigenvectors of A and thereby reveals the internal mathematical and physical structure of the linear system. This means one will be able to predict the evolution $\vec{u}(t)$ for all possible initial value $\vec{u}(t=0) = \vec{u}_0$.

A) Alegraic Analysis

The motivation for the eigenvector method comes from the familiar one-dimensional equation

$$\frac{du}{dt} = Au\,,$$

It solution is

$$u(t) = f(t)\,c$$

where c is an arbitrary constant and

$$f(t) = e^{At}.$$

Q: How does one apply this method if A is $n \times n$?

A: We are looking for an n-dimensional vector $\vec{u}$. This suggests that we look for solutions having the form

$$\vec{u}(t) = f(t)\vec{x}$$

where $\vec{x}$ is a constant vector. Physically this means that all components of $\vec{u}(t)$ have the same time dependence.

Inserting this type of a solution into the equation

$$\frac{d\vec{u}}{dt} = A\vec{u}$$

yields

$$A f \vec{x} = \frac{df}{dt}\vec{x}$$

or

$$\underbrace{A\vec{x} = \frac{1}{f}\frac{df}{dt}}_{\text{independent of time}}\vec{x}$$

This equation implies:

(i) $\frac{1}{f}\frac{df}{dt} = \text{constant} \equiv \lambda \Rightarrow f = e^{\lambda t}$

(ii) $(A-\lambda I)\vec{x} = 0$

One must find values λ for which $N(A-\lambda I) \neq \{\vec{0}\}$

For each such value of λ the solution is

$$\vec{u}(t) = e^{\lambda t}\vec{x} \quad \text{where} \quad \vec{x} \in N(A-\lambda I)$$

B) Comments

(i) $\vec{u}(t) = e^{\lambda t}\vec{x}$ is called an eigenstate. (or an eigen mode if λ is complex)

Comment (continued)

(ii) The linearity of the system implies that the

$$\begin{pmatrix}\text{general}\\ \text{solution}\end{pmatrix} = c_1 e^{\lambda_1 t}\vec{x}_1 + c_2 e^{\lambda_2 t}\vec{x}_2 + c_n e^{\lambda_n t}\vec{x}_n$$

whenever there are n distinct eigenvalues

(iii) One obtains such a solution only for a time invariant system

(iv) For λ to be an eigenvalue of A the following conditions are equivalent

(1) $Ax = \lambda x$ has a nontrivial sol'n

(2) $A-\lambda I$ is singular (columns form a linearly dependent set)

(3) $A-\lambda I$ has linearly dependent rows

(4) $\det|A-\lambda I| = 0$

Example

$\frac{d\vec{u}}{dt} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \vec{u}$ with $\vec{u}(0) = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$

Algebraic analysis

Step 1. $AX = \lambda X$

$\det|A - \lambda I| = 0 \Rightarrow \lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda - 1) = 0$

$\lambda_1 = -1$; compute $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 2$; compute $\vec{x}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

Step 2. Write down the general solution

$$\vec{u} = c_1 e^{\lambda_1 t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{\lambda_2 t}\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Step 3. Write down the solution which satisfies the initial conditions

$$\vec{u}(0) = \begin{bmatrix} 8 \\ 5 \end{bmatrix} \Rightarrow \vec{u}(t) = 3e^{\lambda_1 t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1e^{\lambda_2 t}\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{-t} + 5e^{2t} \\ 3e^{-t} + 2e^{2t} \end{bmatrix}$$

C) Geometrical Analysis

The geometrical method applies to linear as well as to non-linear systems.

Consider

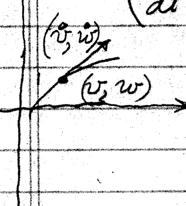
$$\left.\begin{aligned} \frac{dv}{dt} &= f(v,w) = 4v - 5w \\ \frac{dw}{dt} &= g(v,w) = 2v - 3w \end{aligned}\right\} (*)$$

Here

$\big(v(t), w(t)\big)$ = position in $V = R^2$ as a fn of time.

$\left(\frac{dv}{dt}, \frac{dw}{dt}\right)$ = velocity at point $(v(t), w(t))$

= tangent to curve



$$\frac{dw}{dv} = \frac{dw/dt}{dv/dt} = \frac{g(v,w)}{f(v,w)} = \text{slope of tangent at } (v,w)$$

Problem

a) Sketch the phase portrait of Eq. (*) for which one has

$$\frac{dw}{dv} = \frac{2v - 3w}{4v - 5w}.$$

b) Exhibit the tangent at strategic pts.

Solution:

$$\frac{d}{dt}\begin{bmatrix} v \\ w \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}}_{A}\begin{bmatrix} v \\ w \end{bmatrix}$$

A : $\lambda_1 = -1 \Rightarrow \begin{bmatrix} v \\ w \end{bmatrix} \propto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\lambda_2 = 2 \Rightarrow \begin{bmatrix} v \\ w \end{bmatrix} \propto \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ 32.9

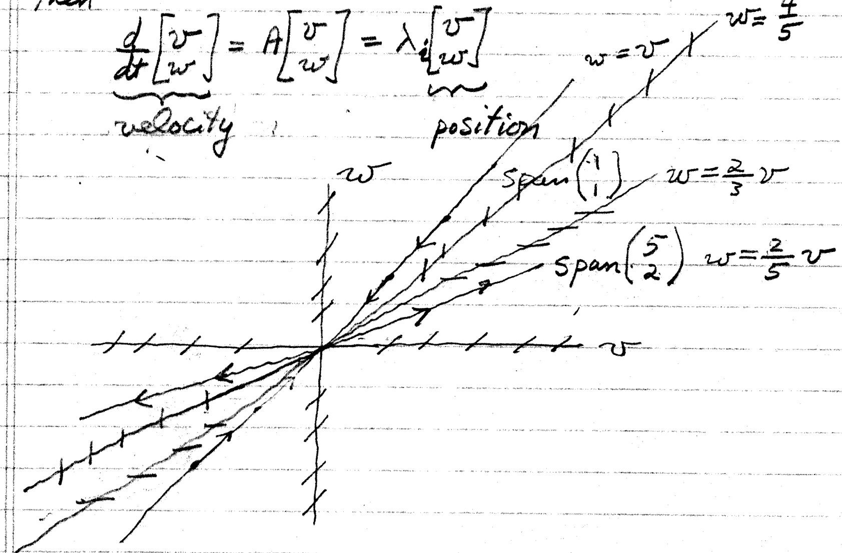
What is the phase space diagram in the v-w plane?

$\frac{w}{v} = \frac{2}{5}$

Let $\begin{bmatrix} v \\ w \end{bmatrix} \in \mathcal{N}(A - \lambda_i I)$

Then

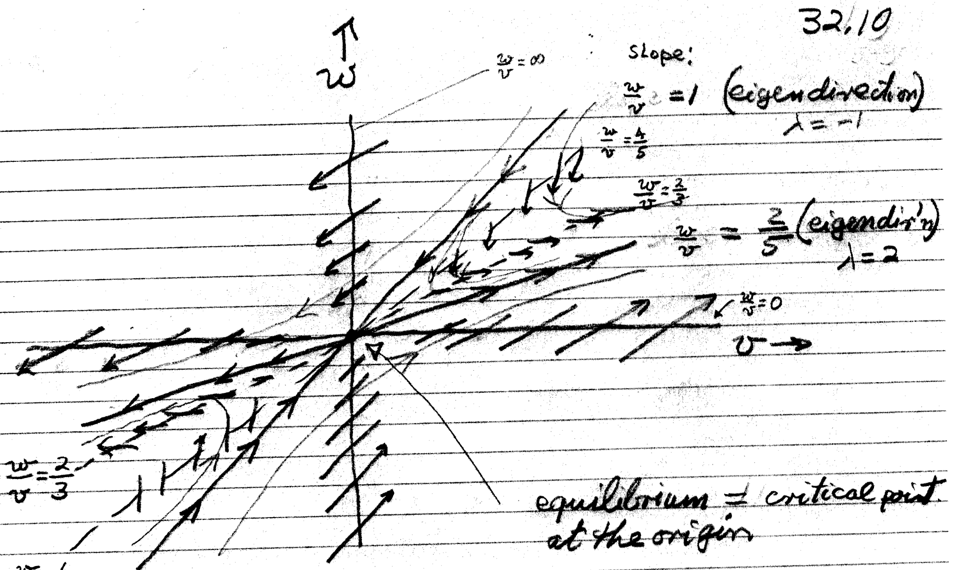
$$\underbrace{\frac{d}{dt}\begin{bmatrix} v \\ w \end{bmatrix}}_{\text{velocity}} = A\begin{bmatrix} v \\ w \end{bmatrix} = \lambda_i \underbrace{\begin{bmatrix} v \\ w \end{bmatrix}}_{\text{position}}$$



$$\frac{dw}{dv} = \frac{2v - 3w}{4v - 5w},$$

$w = 0 \Rightarrow \frac{dw}{dv} = \frac{1}{2}$ $\qquad w = \frac{2}{3} v \Rightarrow \frac{dw}{dv} = 0$

$v = 0 \Rightarrow \frac{dw}{dv} = \frac{3}{5}$ $\qquad w = \frac{4}{5} w \Rightarrow \frac{dw}{dv} = \infty$



$\frac{dw}{dv} = 0$: $\frac{w}{v} = \frac{2}{3}$

$\frac{dw}{dv} = \infty$: $\frac{w}{v} = \frac{4}{5}$

$\downarrow$

$v = \frac{5}{4} w \quad \frac{dw}{dt} = (2 \cdot \frac{5}{4} - 3) w$

$= (\frac{5}{2} - 3) w = -\frac{1}{3} w$

D) Recommendation:

Use "quiver" command in Matlab

III. DIAGONOL FORM OF A MATRIX

Read Strang 5.2

The essential properties of a $n \times n$ matrix with a set of n linearly independent eigen vectors are expressed in terms of its eigenvalues and the transition matrix formed from these eigen vectors. This is expressed by the following

Proposition

Let $A: R^n \to R^n$ be an $n \times n$ matrix
Let A have a set of n lin. indep. eigen vectors
Let S be the transition matrix formed from these eigenvalues.

Then

$$S^{-1} A S = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} .$$

One says that A can be diagonalized

proof:

Consider the equations

$$A \vec{x}_i = \lambda_i \vec{x}_i \qquad i = 1, \ldots, n$$

satisfied by the given eigenvectors.

One has

$$A \begin{bmatrix} \vdots & & \vdots \\ \vec{x}_1 & \cdots & x_n \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & & \vdots \\ \lambda_1 \vec{x}_1 & \cdots & \lambda_n \vec{x}_n \\ \vdots & & \vdots \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \vdots & & \vdots \\ x_1 & \cdots & x_n \\ \vdots & & \vdots \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\Lambda}$$

or

$$A \; S = S \Lambda \qquad \text{where } S \text{ is invertible (why?)}$$

Thus $\boxed{S^{-1} A S = \Lambda}$ or $\boxed{A = S \Lambda S^{-1}}$

A is diagonalizable by means of a similarity transformation.

Comments:

1. It is easy to compute powers of a diagonalizable matrix:

$$A^m = (S \Lambda S^{-1}) \cdots (S \Lambda S^{-1})$$
$$= S \Lambda \cdots \Lambda S^{-1}$$
$$\boxed{A^m = S \Lambda^m S^{-1}}$$

2. One can compute matrix power series: Consider a function defined by a convergent power series

$$g(x) = a_0 + a_1 x + a_2 x^2 + \cdots \; ; \text{e.g. } e^x = 1 + x + \frac{x^2}{2} + \cdots$$

Then $g(A)$ is defined and convergent in an appropriate sense:

$$e^{At} = I + At + A^2\frac{t^2}{2} + \cdots$$

$$= SIS^{-1} + S\Lambda S^{-1}t + S\Lambda^2 S^{-1}\frac{t^2}{2!} + \cdots$$

$$= S\left(I + \Lambda t + \frac{\Lambda^2 t^2}{2!} + \cdots\right)S^{-1}$$

$$= S \begin{bmatrix} e^{\lambda_1 t} & & O \\ & e^{\lambda_2 t} & \\ O & & \ddots \; e^{\lambda_n t} \end{bmatrix} S^{-1}$$

$$= S\, e^{\Lambda t} S^{-1}$$

Thus

$$\boxed{g(A) = S\, g(\Lambda) S^{-1}}$$

3. Consider the secular (a.k.a. characteristic) polynomial

$$\det|A - \lambda I| = a_0 + a_1(-\lambda) + \cdots + a_n(-\lambda)^n$$
$$= (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda) \equiv f(\lambda)$$

The secular pol'l in the powers of A vanishes:

$$f(A) = S^{-1} f(\Lambda) S =$$

$$= S^{-1}\begin{bmatrix} (\lambda_1-\lambda_1)(\lambda_2-\lambda_1)\cdots(\lambda_n-\lambda_1) & & O \\ O & \ddots & (\lambda_1-\lambda_n)\cdots(\lambda_n-\lambda_n) \end{bmatrix} = S^{-1}\begin{bmatrix} 0 & & O \\ & 0 & \\ O & & 0 \end{bmatrix} S = [O]_{\text{matrix}}$$

Thus

$$\boxed{f(A) = a_0 + a_1(-A) + a_2(-A)^2 + \cdots a_{n-1}(-A)^{n-1} + (-A)^n = \text{zero matrix}}$$

This illustrates the Cayley-Hamilton theorem:

A matrix satisfies its characteristic equation

Here the coefficients

$$a_0 = f(0) = \lambda_1\lambda_2\cdots\lambda_n = \det A$$

$$a_{n-1} = \lambda_1 + \cdots + \lambda_n = \sum_{i=1}^{n} A_{ii} = \operatorname{trace} A$$

depend only on the eigenvalues of A.