LECTURE 32

Time Invariant ("Autonomous") Linear Systems.

I. Solution via Taylor Series

II. Solution via Eigenvalues & Eigenvectors

A) Algebraic Analysis

B) Comments

C) Geometric Analysis

D) Recommendation.

III. Diagonal Form of a Matrix

TIME INVARIANT (AUTONOMOUS)

LINEAR SYSTEMS

Consider the system of differential equations with constant coefficients

$$\frac{d\vec{u}(t)}{dt} = A\,\vec{u}(t) \qquad A: V = R^n \to V = R^n$$

Find $\vec{u}(t)$ subject to the initial condition $\vec{u}(0) = \vec{u}_0$

Comment:
Here we develop the mathematical theory for a linear system represented by a constant $n \times n$ matrix A.

There are two mathematical ways of solving this dynamical system problem, namely via

I. Taylor series method

II. Eigenvector method

I. Taylor Series Method

This method can be used for any matrix A as long as it is a constant matrix. Indeed one has

$$\frac{du(t)}{dt} = A\,u(t)$$

$$\frac{d^2u(t)}{dt^2} = A\frac{du}{dt} = A^2 u(t)$$

$$\frac{d^3u}{dt^3} = A^3 u$$

The Taylor series expansion around $t=0$ yields

$$\vec{u}(t) = u(0) + \left.\frac{du}{dt}\right|_0 t + \left.\frac{d^2u}{dt^2}\right|_0 \frac{t^2}{2!} + \cdots$$

$$= \underbrace{\left[I + At + \frac{1}{2!}A^2t^2 + \cdots \right]}_{\equiv\, e^{At}} \vec{u}(0)$$

whenever the series converges.

Thus one has

$$\boxed{\vec{u}(t) = e^{At}\,\vec{u}_0}$$

and

$$\vec{u}(t_1 + t_2) = e^{At_1} e^{At_2} \vec{u}_0 = e^{A(t_1+t_2)}\vec{u}_0$$

One says that e^{At} is the time xlation operator of the system. Aside from the fact that A is said to be the generator of time translations of the system, there is not too much one can say about it without examining the structure of the $n \times n$ matrix A. This deficiency is remedied by the

II. Eigenvector Method

The virtue of this method is that it finds and uses the eigenvalues and eigenvectors of A and thereby reveals the internal mathematical and physical structure of the linear system. This means one will be able to predict the evolution $\vec{u}(t)$ for all possible initial value $\vec{u}(t=0) = \vec{u}_0$.

A) Alegraic Analysis

The motivation for the eigenvector method comes from the familiar one-dimensional equation

$$\frac{du}{dt} = Au ,$$

It solution is

$$u(t) = f(t)\, c$$

where c is an arbitrary constant and

$$f(t) = e^{At}.$$

Q: How does one apply this method if A is $n \times n$?

A: We are looking for an n-dimensional vector $\vec{u}$. This suggests that we look for solutions having the form

$$\vec{u}(t) = f(t)\vec{x}$$

where $\vec{x}$ is a constant vector. Physically this means that all components of $\vec{u}(t)$ have the same time dependence.

Inserting this type of a solution into the equation

$$\frac{d\vec{u}}{dt} = A\vec{u}$$

yields

$$A f \vec{x} = \frac{df}{dt}\vec{x}$$

or

$$\underbrace{\underset{\sim}{A\vec{x}} = \frac{1}{f}\frac{df}{dt}\underset{\sim}{\vec{x}}}_{\text{independent of time}}$$

This equation implies:

(i) $\frac{1}{f}\frac{df}{dt} = \text{constant} \equiv \lambda \Rightarrow f = e^{\lambda t}$

(ii) $(A - \lambda I)\vec{x} = 0$

One must find values λ for which $\mathcal{N}(A-\lambda I) \neq \{\vec{0}\}$

For each such value of λ the solution is

$$\vec{u}(t) = e^{\lambda t}\vec{x} \quad \text{where} \quad \vec{x} \in \mathcal{N}(A - \lambda I)$$

B) <u>Comments</u>

(i) $\vec{u}(t) = e^{\lambda t}\vec{x}$ is called an <u>eigen state</u>. (or an <u>eigen mode</u> if λ is complex)

Comment (continued)

(ii) The linearity of the system implies that the

a) $$\boxed{\begin{pmatrix}\text{general}\\ \text{solution}\end{pmatrix} = c_1 e^{\lambda_1 t}\vec{x}_1 + c_2 e^{\lambda_2 t}\vec{x}_2 + \cdots + c_n e^{\lambda_n t}\vec{x}_n}$$

b) $c_1, \ldots, c_n$ are determined by the initial condxn $\vec{u}(0) = \sum_{k=1}^{n} c_n \vec{x}_k$

whenever there are n distinct eigenvalues

(iii) One obtains such a solution only for a time invariant system

(iv) For λ to be an eigenvalue of A the following conditions are equivalent

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(1) $Ax = \lambda x$ has a nontrivial sol'n

(2) $A - \lambda I$ is singular (columns form a linearly dependent set)

(3) $A - \lambda I$ has linearly dependent rows

(4) $\det|A - \lambda I| = 0$

Example

$$\frac{d\vec{u}}{dt} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \vec{u} \quad \text{with } \vec{u}(0) = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

Algebraic analysis

Step 1. $AX = \lambda X$

$$\det|A - \lambda I| = 0 \implies \lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda - 1) = 0$$

$\lambda_1 = -1$; compute $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 2$; compute $\vec{x}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

Step 2. Write down the general solution

$$\vec{u} = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Step 3. Write down the solution which satisfies the initial conditions

$$\vec{u}(0) = \begin{bmatrix} 8 \\ 5 \end{bmatrix} \Rightarrow \vec{u}(t) = 3e^{\lambda_1 t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 e^{\lambda_2 t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{-t} + 5e^{2t} \\ 3e^{-t} + 2e^{2t} \end{bmatrix}$$

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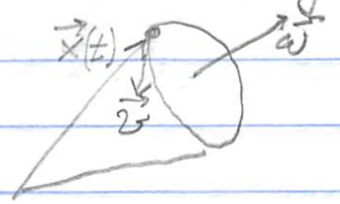
Consider the position-velocity relation



$$\vec{v} = \omega \times \vec{X}$$

$$[\vec{v}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ ← SKIP THIS IN CLASS

$$[\vec{v}] = [\omega][\vec{X}]$$

Q: What are the corresponding particle trajectories?

A: They are the solutions to

$$\underbrace{\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{[\vec{v}]} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{[\vec{x}]}$$

This system can be solved using the pwr series method or the eigenvalue/eigenvector method.

TRY IT FOR $\vec{\omega} = 0\vec{i} + 0\vec{j} + \omega_3 \vec{k}$!

c) Geometrical Analysis

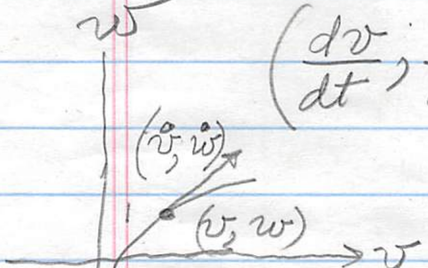
The geometrical method applies to linear as well as to non-linear systems.

Consider

$$\left.\begin{aligned}\frac{dv}{dt} &= f(v,w) = 4v - 5w \\ \frac{dw}{dt} &= g(v,w) = 2v - 3w\end{aligned}\right\} (*)$$

Here

$\big(v(t), w(t)\big)$ = position in $V = R^2$ as a fn of time.



$\left(\frac{dv}{dt}, \frac{dw}{dt}\right)$ = velocity at point $(v(t), w(t))$

= tangent to curve

At each pt of the (v,w) plane $\quad \frac{dw}{dv} = \frac{dw/dt}{dv/dt} = \frac{g(v,w)}{f(v,w)}$ = slope of tangent at (v,w)

Problem

a) Sketch the phase portrait of Eq. (*) for which one has

$$\frac{dw}{dv} = \frac{2v-3w}{4v-5w}.$$

b) Exhibit the tangent at strategic pts.

Solution:

$$\frac{d}{dt}\begin{bmatrix} v \\ w \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -3 \\ 4 & -5 \end{bmatrix}}_{A}\begin{bmatrix} v \\ w \end{bmatrix}; \quad \frac{dv}{dt} = 2v - 3w, \quad \frac{dw}{dt} = 4v - 5w$$

A ! $\lambda_1 = -1 \Rightarrow \begin{bmatrix} v \\ w \end{bmatrix} \propto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\lambda_2 = 2 \Rightarrow \begin{bmatrix} v \\ w \end{bmatrix} \propto \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

What is the phase space diagram in the v-w plane?

$\frac{w}{v} = \frac{2}{5}$

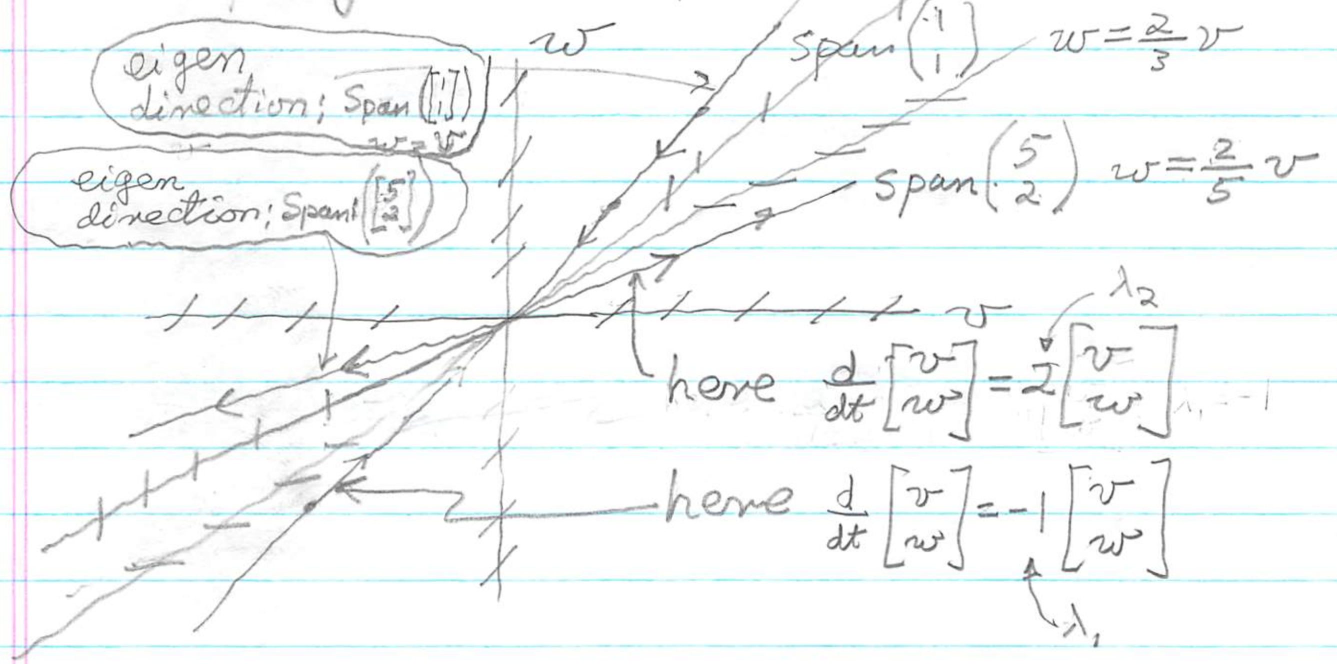
Let $\begin{bmatrix} v \\ w \end{bmatrix} \in \mathcal{N}(A - \lambda_i I)$

Then

$$\underbrace{\frac{d}{dt}\begin{bmatrix} v \\ w \end{bmatrix}}_{\text{velocity}} = A\begin{bmatrix} v \\ w \end{bmatrix} = \lambda_i \underbrace{\begin{bmatrix} v \\ w \end{bmatrix}}_{\text{position}}$$

eigen $w = v$

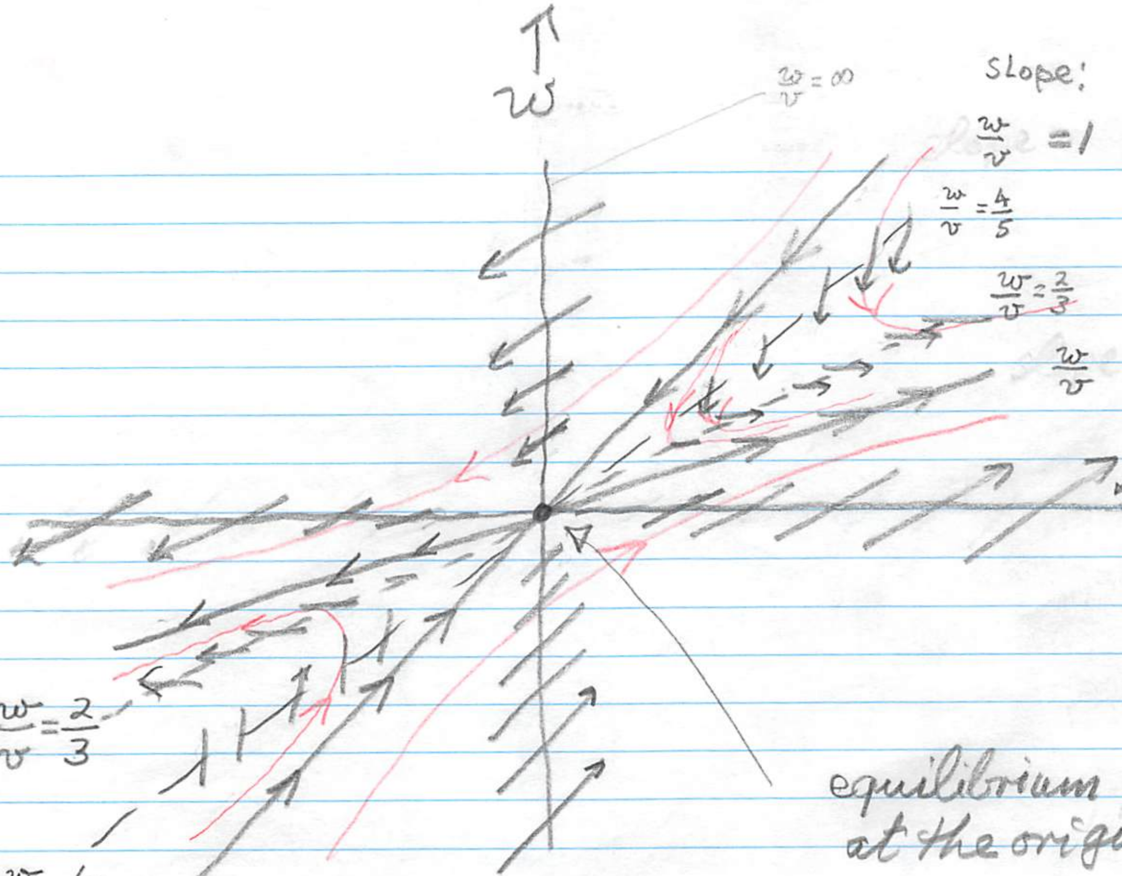
$w = \frac{4}{5}$



$$\frac{dw}{dv} = \frac{2v - 3w}{4v - 5w};$$

$w = 0 \Rightarrow \frac{dw}{dv} = \frac{1}{2}$ $\quad$ $w = \frac{2}{3} v \Rightarrow \frac{dw}{dv} = 0$

$v = 0 \Rightarrow \frac{dw}{dv} = \frac{3}{5}$ $\quad$ $w = \frac{4}{5} w \Rightarrow \frac{dw}{dv} = \infty$



Slope:

$\frac{w}{v}=1$ (eigendirection) $\lambda=-1$

$\frac{w}{v}=\frac{2}{5}$ (eigendir'n) $\lambda=2$

$\frac{dv}{dv}=0$; $\frac{w}{v}=\frac{2}{3}$

equilibrium = critical point at the origin

$\frac{dw}{dv}=\infty$: $\frac{w}{v}=\frac{4}{5}$

$v=\frac{5}{4}w \quad \frac{dw}{dt}=\left(2\cdot\frac{5}{4}-3\right)w$

$=\left(\frac{5}{2}-3\right)w=-\frac{1}{3}w.$

D) <u>Recommendation:</u>

Use "quiver" command in Matlab