APPENDIX A (FOR LECTURE 33)

Linear independence of the set of eigenvectors of a matrix with distinct eigenvalues.

~~Proof of~~ Theorem 33.1

Proof of the linear independence of the set of n eigenvectors having n distinct eigenvalues.

Step I

Consider

$$c_1\vec{e}_1 + c_2\vec{e}_2 = \vec{0} \qquad (1)$$

Apply A to both sides and obtain

$$c_1\lambda_1\vec{e}_1 + c_2\lambda_2\vec{e}_2 = 0$$

Multiply Eq.(1) by λ_2 and subtract. One obtains

$$c_1(\lambda_2-\lambda_1)\vec{e}_1 = 0$$

By hypothesis $\lambda_2 \neq \lambda_1$. Thus

$$c_1 = 0.$$

Introduce this result into Eq.(1), one finds

$$c_2\vec{e}_2 = \vec{0}$$

Thus $c_2 = 0$. It follows that the set of eigenvectors in Eq.(1),

$$\boxed{\{\vec{e}_1, \vec{e}_2\} \text{ is a linearly independent set.}}$$

Step II

One now repeats this line of reasoning by considering the linear combination

$$c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 = \vec{0}$$

to conclude that

$$\{e_1, e_2, e_3\} \text{ is a l.i. set.}$$

Step III. More generally one assumes that

$$\{e_1, e_2, \cdots, e_k\} \text{ is a lin. indep. set} \qquad (2)$$

and considers

$$c_1\vec{e}_1 + \cdots + c_k\vec{e}_k + c_{k+1}\vec{e}_{k+1} = \vec{0} \qquad (3)$$

in order to show that

$$\{e_1, e_2, \cdots, e_k, e_{k+1}\} \text{ is a lin. indep. set.}$$

This is certainly true for $k=1$ and $k=2$.

Step IV.

Apply A to Eq.(3) and obtain

$$c_1\lambda_1\vec{e}_1 + \cdots + c_k\lambda_k\vec{e}_k + c_{k+1}\lambda_{k+1}\vec{e}_{k+1} = \vec{0}. \qquad (4)$$

Mply Eq.(3) by λ_{k+1}, subtract the result from Eq.(4) and obtain

$$c_1(\lambda_1-\lambda_{k+1})\vec{e}_1 + \cdots + c_k(\lambda_k-\lambda_{k+1})\vec{e}_k = \vec{0}$$

By hypothesis $\lambda_1 \neq \lambda_{k+1}, \cdots, \lambda_k \neq \lambda_{k+1}$. Thus Eq.(2) implies

$$c_1 = c_2 = \cdots = c_k = 0.$$

Introduce this result into Eq. (3); thus

$$c_{k+1}\vec{e}_{k+1}=0$$

Consequently, all coefficients vanish:

$$c_1=\cdots=c_k=c_{k+1}=0$$

This means that

$\{e_1, e_2, \ldots, e_k, e_{k+1}\}$ is also a lin. indep. set,

Step V

Repeat Steps III and IV until one has exhausted all n eigenvalues.

The result is that

$\{\vec{e}_1, \ldots, \vec{e}_n\}$ is a lin. indep. set.