LECTURE 34

I.) DECOUPLING A DIAGONALIZABLE SYSTEM.

II.) LINEAR SYSTEM WITH A DEFECTIVE GENERATOR.

III.) HERMETIAN ADJOINT

A) Basis Independent Definition

B) Matrix Definition

C) Matrix Element Definition

IV.) HERMETIAN OPERATOR

Four of Its Properties
Suppose we have a time-invariant linear system whose evolving state $\mathbf{u}(t)$ is governed by the equation \[ \frac{d\mathbf{u}(t)}{dt} = A \mathbf{u}(t) \] with a generating matrix $A$ which has $n$ distinct eigenvalues. In that case $A$ is diagonalizable (Theorem 35.1 on page 35.2) and one has from page 35.2 \[ A = S \Lambda S^{-1}. \]

Introduce this expression into Eq. (5), one obtains \[ \frac{d\mathbf{u}(t)}{dt} = S \Lambda S^{-1} \mathbf{u}(t) \] Multiply on the left by $S^{-1}$, which is independent of time, and introduce the new representation of the evolving state, namely \[ \mathbf{\tilde{u}}(t) = S^{-1} \mathbf{u}(t). \]

The governing equation becomes \[ \frac{d\mathbf{\tilde{u}}(t)}{dt} = \Lambda \mathbf{\tilde{u}}(t). \] or \[ \begin{align*}
\frac{d}{dt} & \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} \\
\frac{d}{dt} u_n(t) & = \lambda_n u_n(t)
\end{align*} \]

This is a remarkable result. The change of dependent variable \[ \mathbf{\tilde{u}}(t) = S \mathbf{\tilde{u}}(t) \] has decoupled the system of $n$
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equations, Eq. (*) on page 34.1, into a set of separate equations for each of its respective degrees of freedom.

One therefore has $n$ solutions

$$\ddot{v}_i(t) = c_i e^{\lambda_i t}, \quad i = 1, \ldots, n$$

Thus one has

$$\tilde{u}(t) = S \begin{bmatrix} e^{\lambda_1 t} \\ 0 \\ \vdots \\ 0 \\ e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

or

$$\tilde{u}(t) = c_1 S \begin{bmatrix} e^{\lambda_1 t} \\ \vdots \\ 0 \end{bmatrix} + c_2 S \begin{bmatrix} 0 \\ \vdots \\ e^{\lambda_2 t} \end{bmatrix} + \ldots + c_n S \begin{bmatrix} 0 \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

The general solution is a linear superposition of the respective degrees of freedom, each one of which characterized by a distinct temporal evolution characterized by its own $e^{\lambda_i t}, \quad i = 1, \ldots, n$.

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LINEAR SYSTEM WITH A DEFECTIVE GENERATOR

Suppose we have a time invariant linear system whose evolving state $\tilde{u}(t)$ is governed by the equation

$$\frac{d\tilde{u}}{dt} = A \tilde{u}(t)$$

whose generating matrix has an eigenvalue $\lambda$ and has corresponding $k$ generalized eigenvectors

$$X_1, X_2, \ldots, X_k$$

of order $1, 2, \ldots, k$ respectively

$$(A - \lambda)X_k = 0 \quad \text{but} \quad (A - \lambda)^{k-1}X_k \neq 0$$

The corresponding solutions of the governing equation are

INSERT DEFIN from page 34.14 in APP.
\[ \tilde{u}_2(t) = e^{\lambda t} \tilde{x}_2 \]
\[ \tilde{u}_k(t) = e^{\lambda t} \left[ I + (A_2 - \lambda) t \frac{t}{2!} + \cdots + (A_2 - \lambda)^{k-1} t^{k-1} \frac{(k-1)!}{k!} \right] \tilde{x}_k \]

and the general solution is a linear combination of these:

\[ \tilde{u}(t) = c_1 \tilde{u}_1(t) + c_2 \tilde{u}_2(t) + \cdots + c_k \tilde{u}_k(t) \]

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**III. HERMETIAN ADJOINT**: 34.6

A. If a matrix \( A \) acts on a vector space with an inner product structure, then this inner product,

\[ \langle x, y \rangle, \quad x, y \in V \]

establishes a correspondence between \( A \) and its Hermetian adjoint by means of the following Definition (basis independent def'n)

Let \( A : V \rightarrow V = \text{complex inner product space} \)

Then the Hermetian adjoint \( A^H \) of \( A \) is defined by

\[ \langle A^H x, y \rangle = \langle x, Ay \rangle \quad \forall x, y \in V \]
### B. Definition (Matrix version)

1. Let us define the standard complex inner product using the notation,

\[
\begin{align*}
\mathbf{y} & = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \\
\mathbf{x}^T & = [x_1 \ldots x_n]^T \\
\mathbf{x}^H & = [\overline{x}_1 \ldots \overline{x}_n] \\
\langle \mathbf{x}, \mathbf{y} \rangle & = \overline{x}_1 y_1 + \ldots + \overline{x}_n y_n = \mathbf{x}^T \mathbf{y} = \mathbf{x}^H \mathbf{y}
\end{align*}
\]

2. Apply this inner product to the definition of the Hermitian adjoint of \( \mathbf{A} \)

\[
\begin{align*}
\mathbf{x}^H \mathbf{A} \mathbf{y} &= \langle \mathbf{x}, \mathbf{A} \mathbf{y} \rangle = \langle \mathbf{A}^H \mathbf{x}, \mathbf{y} \rangle = (\mathbf{A}^H \mathbf{x})^H \\
&= (\mathbf{A}^H \mathbf{x})^T \mathbf{y} = \mathbf{x}^T (\mathbf{A}^H)^T \mathbf{y} \\
&= \mathbf{x}^H (\mathbf{A}^H)^T \mathbf{y} \quad \forall \mathbf{x}, \mathbf{y}
\end{align*}
\]

3. Conclusion

\[
\begin{align*}
\mathbf{A}^H &= (\mathbf{A}^H)^T \\
\mathbf{A}^H &= \mathbf{A}^T
\end{align*}
\]
Comments

1. \( A_{ij} \) are the matrix elements of \( A \) relative to the given basis.
   and
   \( A^{H}_{ij} \) are the matrix elements of \( A^{H} \) relative to the given basis.

2. The matrix elements of \( A^{H} \) are obtained from \( A \) by taking the transpose and complex conjugate of the matrix elements of \( A \).
   \[
   A^{H}=A
   \]
   or
   \[
   \langle Ax,y \rangle = \langle x, Ay \rangle \quad \forall x,y
   \]
   or
   \[
   A_{ij} = \overline{A_{ji}}
   \]

Examples:

(i) \[
\begin{bmatrix}
2 & 3-3i \\
3+i & 5
\end{bmatrix}
\]
   is Hermitian on \( \mathbb{C}^{2} \) wrt standard inner product.

(ii) \[
\begin{bmatrix}
1 & 3 \\
3 & 2
\end{bmatrix}
\]
   is Hermitian on \( \mathbb{R}^{2} \) wrt standard inner product.
### Property 3

\[ \mathbf{A}^H = \mathbf{A} \] implies eigenvectors of different eigenvalues are orthogonal. Indeed on has

\[ \mathbf{A} \mathbf{x}_1 = \lambda_1 \mathbf{x}_1 \Rightarrow \langle \mathbf{x}_2, \mathbf{A} \mathbf{x}_1 \rangle = \langle \mathbf{x}_2, \lambda_1 \mathbf{x}_1 \rangle \]

\[ \langle \mathbf{A} \mathbf{x}_2, \mathbf{x}_1 \rangle = \lambda_1 \langle \mathbf{x}_2, \mathbf{x}_1 \rangle \]

Thus \( (\lambda_2 - \lambda_1) \langle \mathbf{x}_2, \mathbf{x}_1 \rangle = 0 \)

\( \lambda_2 \) is real. Also \( \lambda_2 \neq \lambda_1 \). Thus

\[ \langle \mathbf{x}_2, \mathbf{x}_2 \rangle = 0 \]

### Property 4

\[ \mathbf{A}^H = \mathbf{A} \] implies \( \exists \) a unitary matrix \( \mathbf{U} \) such that

\[ \mathbf{U}^{-1} \mathbf{A} \mathbf{U} = \mathbf{U}^H \mathbf{A} \mathbf{U} = \mathbf{\Lambda} \]

(\( \mathbf{\Lambda} \) is diagonal)

\[ \mathbf{U}^{-1} = \mathbf{U}^H \]

\[ \mathbf{U}^H \mathbf{U} = \mathbf{I} \]

\[ \mathbf{U}^{-1} \mathbf{A} \mathbf{U} = \mathbf{\Lambda} \]
Indeed, we have

\[
\begin{bmatrix}
  x_1 & \cdots & x_n
\end{bmatrix}
\]

without loss of generality, once has

\[
\langle x_i, x_j \rangle = \delta_{ij} \quad \text{(from prop. 2)}
\]

Equation (*) implies

\[
U U^H = \begin{bmatrix}
  0 & 1
\end{bmatrix} = I
\]

We have

\[
U^H = \begin{bmatrix}
  -x_i^H \\
  \begin{bmatrix}
  -x_i^H \\
  \end{bmatrix}
\end{bmatrix}
\]

\[
U^H U = \begin{bmatrix}
  0 & 1
\end{bmatrix} = I
\]