LECTURE 34

I.) DECOUPLING A DIAGONALIZABLE SYSTEM.

II.) LINEAR SYSTEM WITH A DEFECTIVE GENERATOR.

For next Lecture

III.) HERMETIAN ADJOINT

A) Basis Independent Definition

B) Matrix Definition

c) Matrix Element Definition

IV HERMETIAN OPERATOR

Four of Its Properties

I.) DECOUPLING A DIAGONALIZABLE SYSTEM INTO ITS DISTINCT DEGREES of FREEDOM.

Suppose we have a time invariant linear systems whose evolving state $\vec{u}(t)$ is governed by the equation

$$\frac{d\vec{u}(t)}{dt} = A\,\vec{u}(t) \qquad (*)$$

with a generating matrix A which has n distinct eigenvalues. In that case A is diagonalizable (Theorem 33.1 on page 33.12) and one has from page 33.2

$$A = S\Lambda S^{-1}.$$

Introduce this expression into Eq. (*), one obtains

$$\frac{d\vec{u}(t)}{dt} = S\Lambda S^{-1}\vec{u}(t)$$

Multiply on the left by S^{-1}, which is independent of time, and introduce the

new representation of the evolving state, namely

$$\vec{v}(t) = S^{-1}\vec{u}(t).$$

The governing equation becomes

$$\frac{d\vec{v}(t)}{dt} = \Lambda\,\vec{v}(t)$$

$$\frac{d}{dt}\begin{bmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{matrix} v_1(t) \\ \vdots \\ v_n(t) \end{matrix}$$

or

$$\boxed{\begin{aligned} \frac{dv_1(t)}{dt} &= \lambda_1 v_1(t) \\ &\vdots \\ \frac{dv_n(t)}{dt} &= \lambda_n v_n(t) \end{aligned}}$$

This is a remarkable result. The change of dependent variable

$$\vec{u}(t) = S\,\vec{v}(t)$$

has decoupled the system of n

equations, Eq.(*) on page 34.1,

into a set of n separate equations for each of its respective degrees of freedom $v_1(t), \dots, v_n(t)$.

One therefore has n solutions

$$v_i(t) = c_i e^{\lambda_i t} \qquad i = 1, \dots, n$$

Thus one has

$$\vec{u}(t) = S \begin{bmatrix} e^{\lambda_1 t} & & O \\ & \ddots & \\ O & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

or

$$\vec{u}(t) = c_1 S \begin{bmatrix} e^{\lambda_1 t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2 S \begin{bmatrix} 0 \\ e^{\lambda_2 t} \\ \vdots \\ 0 \end{bmatrix} + \dots + c_n S \begin{bmatrix} 0 \\ 0 \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

The general solution is a linear superposition of the respective degrees of freedom each one of which characterized by a distinct temporal evolution characterized by its own $e^{\lambda_i t}$ $i = 1, \dots, n$

II) LINER SYSTEM WITH A DEFECTIVE GENERATOR

Suppose we have a time invariant linear system whose evolving state $\vec{u}(t)$ is governed by the equation

$$\frac{d\vec{u}}{dt} = A\vec{u}(t)$$

whose generating matrix has an eigenvalue λ and has corresponding k generalized eigenvectors

$$\vec{X}_1, \vec{X}_2, \dots, \vec{X}_k$$

of order $1, 2, \dots, k$ respectively

$$(A-\lambda)^k \vec{X}_k = 0 \text{ but } (A-\lambda)^{k-1} \vec{X}_k \neq 0$$

INSERT DEF'N from page ~~34.A4~~ 34. in Appendix

The corresponding solutions of the governing equation are

"

$$\boxed{\begin{aligned} \vec{u}_1(t) &= e^{\lambda t}\vec{x}_1 \\ \vec{u}_2(t) &= e^{\lambda t}\left[I+(A-\lambda)t\right]\vec{x}_2 \\ &\vdots \\ \vec{u}_k(t) &= e^{\lambda t}\left[I+(A-\lambda)t+(A-\lambda)^2\frac{t^2}{2!}+\cdots+(A-\lambda)^{k-1}\frac{t^{k-1}}{(k-1)!}\right]\vec{x}_k \end{aligned}}$$

and the general solution is a linear combination of these;

$$\vec{u}(t) = c_1\vec{u}_1(t) + c_2\vec{u}_2(t) + \cdots + c_k\vec{u}_k(t)$$

III.) HERMETIAN ADJOINT.

A. If a matrix A acts on a vector space with an inner product structure, then this inner product,

$$\langle x, y\rangle \quad , \quad x, y \in V$$

establish a correspondence between A and its "Hermetian adjoint" by means of the following

Definition (Basis independent def'n)

Let $A: V \to V$ = complex inner product space

Then the Hermetian adjoint A^H of A is defined by

$$\boxed{\langle A^H x, y\rangle = \langle x, Ay\rangle \qquad \forall x, y \in V}$$

B. Definition (Matrix version)

a) Let us define the standard complex inner product using the notation

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad x^T = [x_1 \;\cdots\; x_n] \qquad x^H = [\bar{x}_1 \;\cdots\; \bar{x}_n]$$

$$\langle x, y \rangle = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n = \overline{x^T}\, y \equiv x^H y$$

b) Apply this inner product to the definition of the Hermetian adjoint of A

$$x^H A y = \langle x, Ay \rangle = \langle A^H x, y \rangle = (A^H x)^H y$$

$$= \overline{(A^H x)^T}\, y = \overline{x^T (A^H)^T}\, y$$

$$= x^H\, \overline{(A^H)^T}\, y \qquad \forall x, y$$

c) Conclusion

$$\boxed{\begin{aligned} A &= \overline{(A^H)^T} \\ \text{or} \quad A^H &= \overline{A^T} \end{aligned}}$$

C) Definition (Definition in terms of matrix elements)

Let $\{u_i\}$ be a basis for V and evaluate

$$\langle A^H x, y \rangle = \langle x, Ay \rangle = \langle x^i u_i, A\, y^j u_j \rangle$$

$$= \bar{x}^i \underbrace{\langle u_i, A u_j \rangle}\, y^j$$

$$\equiv \bar{x}^i \quad A_{ij} \quad y^j \qquad (\star)$$

$$\langle A^H x, y \rangle = \bar{x}^i \langle A^H u_i, u_j \rangle y^j$$

$$= \bar{x}^i\, \overline{\underbrace{\langle u_j, A^H u_i \rangle}}\, y^j$$

$$\equiv \bar{x}^i \quad \overline{A^H_{ji}} \quad y^j \qquad (\star\star)$$

This holds $\forall x, y \in V$.

Compare $(\star)$ with $(\star\star)$ and obtain

$$\boxed{A_{ij} = \overline{A^H_{ji}}}$$

or

$$A^H_{ij} = \overline{A_{ji}} = \overline{A^T_{ij}}$$

Comments

1. A_{ij} are the <u>matrix elements of A</u> <u>relative to the given basis</u>

and

A^H_{ij} are the matrix elements of A^H relative to the given basis.

2. <u>The matrix elements of A^H are obtained from A by taking the transpose and complex conjugate of the matrix elements of A.</u>

IV HERMETIAN OPERATOR

Definition (Hermetian operator)

An operator $A : V \to V$ is said to be Hermetian with respect to the inner product $\langle \ , \ \rangle$ if

$$\boxed{\begin{array}{l} A^H = A \\ \text{or} \quad \langle Ax, y\rangle = \langle x, Ay\rangle \quad \forall x, y \\ \text{or} \quad A_{ij} = \overline{A_{ji}} \end{array}}$$

Examples:

(i) $\begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$ is Hermetian on C^2 w.r.t. standard inner product

(ii) $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$ is Hermetian on R^2 w.r.t. standard inner product

Comment.
If the inner product is real, then a Hermetian matrix has real entries, and one has

$$A^T = A,$$

i.e the matrix A is symmetric.

V PROPERTIES of HERMETIAN MATRICES

Property 1

$A^H = A$ implies the quadratic form

$$\langle x, Ax\rangle = \bar{x}^i A_{ij} x^j$$

is real. Indeed, one has

$$\overline{\langle x, Ax\rangle} = \langle Ax, x\rangle = \langle A^H x, x\rangle = \langle x, Ax\rangle$$

Property 2
$A^H = A$ implies the eigenvalues of A are real. Indeed, one has

$$Ax = \lambda x \Rightarrow \langle x, Ax\rangle = \langle x, \lambda x\rangle = \lambda \|x\|^2$$

$$\langle x, Ax\rangle \text{ is real} \Rightarrow \lambda \|x\|^2 \text{ is real} \Rightarrow \lambda \text{ is real}$$

Property 3

$A^H = A$ implies eigenvectors of different eigenvalues are orthogonal. Indeed on has

$$Ax_1 = \lambda_1 x_1 \Rightarrow \underbrace{\langle x_2, Ax_1\rangle}_{\underbrace{\langle Ax_2, x_1\rangle}_{\bar{\lambda}_2 \langle x_2, x_1\rangle}} = \underbrace{\langle x_2, \lambda_1 x_1\rangle}_{\lambda_1 \langle x_2, x_1\rangle}$$

Thus $(\bar{\lambda}_2 - \lambda_1)\langle x_2, x_1\rangle = 0$

λ_2 is real. Also $\lambda_2 \neq \lambda_1$. Thus

$$\boxed{\langle x_2, x_1\rangle = 0}$$

Property 4
$A^H = A$ implies $\exists$ a unitary matrix

$$U: \; U^H = U^{-1}; \; U^H U = I$$

such that

$$U^{-1} A U = U^H A U = \Lambda \; (= \text{diagonal matrix})$$

Indeed we have

$$A \underbrace{\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}}_{U} = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Without loss of generality one has

$$\langle x_i, x_j \rangle = \delta_{ij} \qquad \text{(from ppty 3)}$$

$$x_i^H x_j = \delta_{ij} \qquad (*)$$

We have

$$U^H = \begin{bmatrix} - x_1^H - \\ \vdots \\ - x_n^H - \end{bmatrix}$$

Eq. (*) implies

$$U^H U = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = \underline{I}$$