LECTURE 34

I.) DECOUPLING A DIAGONALIZABLE SYSTEM.

II.) DECOUPLING A LINEAR SYSTEM WITH A DEFECTIVE GENERATOR.

III.) HERMETIAN ADJOINT

A) Basis Independent Definition

For Lecture 35

B) Matrix Definition

C) Matrix Element Definition

IV HERMETIAN OPERATOR

Four of Its Properties

I.) DECOUPLING A DIAGONALIZABLE SYSTEM INTO ITS DISTINCT DEGREES of FREEDOM.

Suppose we have a time invariant linear system whose evolving state $\vec{u}(t)$ is governed by the equation

$$\frac{d\vec{u}(t)}{dt} = A\,\vec{u}(t) \qquad (*)$$

with a generating matrix A which has n distinct eigenvalues. In that case A is diagonalizable (Theorem 33.1 on page 33.12) and one has from page 33.2

$$A = S\Lambda S^{-1}.$$

Introduce this expression into Eq. (*), one obtains

$$\frac{d\vec{u}(t)}{dt} = S\Lambda S^{-1}\vec{u}(t)$$

Multiply on the left by S^{-1}, which is independent of time, and introduce the new representation of the evolving state, namely

$$\vec{v}(t) = S^{-1}\vec{u}(t).$$

The governing equation becomes

$$\frac{d\vec{v}(t)}{dt} = \Lambda \vec{v}(t)$$

$$\frac{d}{dt}\begin{bmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_n \end{bmatrix} \begin{bmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{bmatrix}$$

or

$$\boxed{\begin{aligned} \frac{dv_1(t)}{dt} &= \lambda_1 v_1(t) \\ &\vdots \\ \frac{dv_n(t)}{dt} &= \lambda_n v_n(t) \end{aligned}} \qquad (**)$$

This is a remarkable result. The change of dependent variable

$$\vec{u}(t) = S\,\vec{v}(t)$$

has decoupled the system of n

equations, Eq. (*) on page 34.1,

into a set of n separate equations for each of

its respective degrees of freedom $v_1(t), \ldots, v_n(t)$.

One therefore has n solutions

$$v_i(t) = c_i e^{\lambda_i t} \qquad i = 1, \ldots, 2$$

Thus one has

$$\vec{u}(t) = S \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

or

$$\vec{u}(t) = c_1 S \begin{bmatrix} e^{\lambda_1 t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2 S \begin{bmatrix} 0 \\ e^{\lambda_2 t} \\ \vdots \\ 0 \end{bmatrix} + \cdots + c_n S \begin{bmatrix} 0 \\ 0 \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

The general solution is a linear superposition of the respective degrees of freedom each one of which charcterized by a distinct temporal evolution characterized by its own $e^{\lambda_i t}$ $i = 1, \ldots, n$

II.) LINER SYSTEM WITH A DEFECTIVE GENERATOR

Suppose we have a time invariant linear system whose evolving state $\vec{u}(t)$ is governed by the equation

$$\frac{d\vec{u}}{dt} = A\vec{u}(t) \qquad (*)$$

whose generating matrix has an eigen-value λ and has corresponding k generalized eigenvectors

$$\vec{X}_1, \vec{X}_2, \cdots, \vec{X}_k$$

of order $1, 2, \cdots, k$ respectively

$$(A-\lambda)^k \vec{X}_k = 0 \text{ but } (A-\lambda)^{k-1} \vec{X}_k \neq 0$$

The corresponding solutions of the governing equation are

Definition (Generalized eigenvector)

A vector $\vec{x}_k$ is said to be an eigenvector of order k if it has the property that

$$(A-\lambda)^k \vec{x}_k = \vec{0} \quad \text{but} \quad (A-\lambda)^{k-1} \vec{x}_k \neq \vec{0}$$

Thus one has the following chain of generalized eigenvectors

$$(3)\left\{\begin{array}{l}
(A-\lambda I)^0 \vec{x}_k = \vec{x}_k = \text{gen'zed e.v. of order } k, \\
(A-\lambda I)^1 \vec{x}_k \equiv \vec{x}_{k-1} = \text{gen'zed e.v. of order } k-1. \\
(A-\lambda I)^2 \vec{x}_k \equiv \vec{x}_{k-2} = \text{gen'zed e.v. of order } k-2 \\
\quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
(A-\lambda I)^{k-2} \vec{x}_k \equiv \vec{x}_2 = \text{generalized e.v. of order } 2 \\
(A-\lambda I)^{k-1} \vec{x}_k \equiv \vec{x}_1 = \text{gen'd e.v. of order } 1
\end{array}\right.$$

$(A-\lambda I)^k \vec{x}_k = (A-\lambda)\vec{x}_1 = 0$, which means that $\vec{x}_1$ is a standard eigenvector.

The corresponding solutions to the governing Eq.(*) on page 34.4 are

$$\boxed{\begin{aligned} \vec{u}_1(t) &= e^{\lambda t}\vec{x}_1 \\ \vec{u}_2(t) &= e^{\lambda t}\left[I+(A-\lambda)t\right]\vec{x}_2 \\ &\vdots \\ \vec{u}_k(t) &= e^{\lambda t}\left[I+(A-\lambda)t+(A-\lambda)\frac{t^2}{2!}+\cdots+\frac{(A-\lambda)^{k-1}t^{k-1}}{(k-1)!}\right]\vec{x}_k \end{aligned}}$$

and the general solution is a linear combination of these;

$$\vec{u}(t) = c_1\vec{u}_1(t) + c_2\vec{u}_2(t) + \cdots + c_k\vec{u}_k(t)$$