APPENDIX A (FOR LECTURE 34)

Time invariant linear system

$\frac{d\vec{u}}{dt} = A\vec{u}$ with a defective generator A.

Not all matrices posses a sufficient number of eigenvectors. In particular defective matrices do not.

Example

Consider the dynamical system governed by the following differential equation

$$\frac{d}{dt}\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}}_{A}\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \qquad (1)$$

The characteristic equation

$$\det|A-\lambda I| = (\lambda-3)^2 = 0$$

has only one root: $\lambda_1=\lambda_2=3$; for this eigenvalue one has

a) $$[A-3I]\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The exist only one eigenvector, namely

$$\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

b) However, one also has

$$(A-3I)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(A-I)^2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (2)$$

The vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is generalized eigenvector of order 2. This will be discussed below.

As we have seen in Lecture 32 on page 32.2, the solution to an equation like Eq. (1) on page 33.B1 is quite generally

$$\vec{u}(t) = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = e^{At}\begin{bmatrix} x_0^1 \\ x_0^2 \end{bmatrix}$$

In light of Eq.(2) above one writes

$$e^{At} = e^{\lambda t}\, e^{(A-\lambda)t} = e^{\lambda t}\left[I + (A-\lambda)t + \frac{(A-\lambda)^2 t^2}{2!} + \cdots\right]$$

We have $$\begin{bmatrix} x_0^1 \\ x_0^2 \end{bmatrix} = x_0^1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_0^2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Applying the power series expansion e^{At} to this sum, one sees that powers $(A-\lambda)^2$

and higher yield zero; in other words the series truncates. One has

$$e^{At}\begin{bmatrix}x_0^1\\x_0^2\end{bmatrix} = e^{\lambda t}\left[I+(A-\lambda)t+\frac{(A-\lambda)^2}{2}t^2+\cdots\right]\underbrace{\begin{bmatrix}x_0^1\\x_0^2\end{bmatrix}}_{x_0^1\begin{bmatrix}1\\0\end{bmatrix}+x_0^2\begin{bmatrix}0\\1\end{bmatrix}}$$

$$\begin{bmatrix}x^1\\x^2\end{bmatrix} = e^{\lambda t}x_0^1\begin{bmatrix}1\\0\end{bmatrix}+e^{\lambda t}x_0^2\left[I+(A-\lambda)t\right]\begin{bmatrix}0\\1\end{bmatrix}; \quad \lambda=3$$

$$x = e^{3t}x_0^1\begin{bmatrix}1\\0\end{bmatrix}+e^{3t}x_0^2\left(\begin{bmatrix}0\\1\end{bmatrix}+t\begin{bmatrix}1\\0\end{bmatrix}\right)$$

The solution is

$$\boxed{\begin{aligned}\begin{bmatrix}x^1(t)\\x^2(t)\end{bmatrix} &= e^{3t}\begin{bmatrix}1\\0\end{bmatrix}x_0^1+e^{3t}\left(\begin{bmatrix}0\\1\end{bmatrix}+t\begin{bmatrix}1\\0\end{bmatrix}\right)x_0^2\\ \text{or}\quad \begin{bmatrix}x^1(t)\\x^2(t)\end{bmatrix} &= e^{3t}\left\{\begin{bmatrix}x_0^1\\x_0^2\end{bmatrix}+t\begin{bmatrix}1\\0\end{bmatrix}x_0^2\right\}\end{aligned}}$$

This computed solution is mathematized by the following Definition and Theorem

Definition (Generalized eigenvector)

A vector $\vec{x}_k$ is said to be an eigenvector of order k if it has the property that

$$(A-\lambda)^k\vec{x}_k=\vec{0} \quad \text{but} \quad (A-\lambda)^{k-1}\vec{x}_k\neq\vec{0}$$

Thus one has the following chain of generalized eigenvectors

$$(3)\left\{\begin{aligned}(A-\lambda I)^0\vec{x}_k &= \vec{x}_k = \text{gen'ized e.v. of order } k,\\ (A-\lambda I)^1\vec{x}_k &\equiv \vec{x}_{k-1} = \text{gen'ized e.v. of order } k-1.\\ (A-\lambda I)^2\vec{x}_k &\equiv \vec{x}_{k-2} = \text{gen'ized e.v. of order } k-2\\ &\ \,\vdots\\ (A-\lambda I)^{k-1}\vec{x}_k &\equiv \vec{x}_1 = \text{gen'd e.v. of order } 1\end{aligned}\right.$$

$(A-\lambda I)^k\vec{x}_k = (A-\lambda)\vec{x}_1 = 0$, which means that $\vec{x}_1$ is a standard eigenvector.

Thes k generalized eigenvector give rise to the following

Theorem (Indep. set of generalized eigenvectors)

The set of vectors $\vec{X}_1, \vec{X}_2, \dots, \vec{X}_k$ form a linearly independent set.

Proof:
Consider the linear combination

$$c_k \vec{X}_k + c_{k-1}\vec{X}_{k-1} + \dots + c_1 \vec{X}_1 = \vec{0}$$

Applying $(A-\lambda)^{k-1}, (A-\lambda)^{k-2}, \dots, (A-\lambda)^1$ in turn one obtains

$$c_k = 0,\ c_{k-1} = 0,\ \dots,\ c_1 = 0$$

respectively. Thus

$\{\vec{X}_k, \dots, \vec{X}_1\}$ is lin. indep. indeed.

Comment

If $\vec{X}_k$ is a gen. e.v. of order k, then so is

$$\boxed{\beta_k \vec{X}_k + \beta_{k-1}\vec{X}_{k-1} + \dots + \beta_1 \vec{X}_1}$$

The availability of the chain of generalized eigenvectors is the basis for solving the linear system

$$\frac{d\vec{u}(t)}{dt} = A\vec{u}(t). \qquad (4)$$

Proposition

GIVEN: Let λ be an eigenvalue of A
Let $\vec{X}_k$ be a corresponding generalized eigenvector of order k.

CONCLUSION
The corresponding solution to Eq. (4) is

$$\vec{u}(t) = c_1 e^{\lambda t}\vec{X}_1 + c_2 e^{\lambda t}(\vec{X}_2 + t\vec{X}_1) + \dots + c_k e^{\lambda t}\left(\vec{X}_k + t\vec{X}_{k-1} + \dots + \frac{t^{k-1}}{k-1!}\vec{X}_1\right)$$

3.4.7

Proof (by a 2 step construction)

Step 1

For any constant vector $\vec{x}$, including $\vec{x}_k$, one has a solution

$$\vec{u}_k(t) = e^{At}\vec{x}_k$$

$$= e^{\lambda t} e^{(A-\lambda)t}\vec{x}_k$$

$$= e^{\lambda t}\left[I + (A-\lambda)t + \cdots + \frac{(A-\lambda)^{k-1}}{(k-1)!}t^{k-1}\right]\vec{x}_k$$

The series trunkates at the $(k-1)$ st term because

$$(A-\lambda)^k\vec{x}_k = 0.$$

Step 2

One obtains solutions $\vec{u}_{k-1}(t), \cdots, \vec{u}_1(t)$ in an analogous fashion. Because Eq.(4) is a linear system, their linear combination

$$\boxed{\vec{u}(t) = c_k u_k(t) + c_{k-1}u_{k-1}(t) + \cdots + c_1 u_1(t)}$$

is also a solution.