LECTURE 35

I. Hermitian Adjoint

Definition: Basis Independent
Definition: Matrix Version
Definition: Matrix element-wise

II. Hermitian Operators

III. Properties of Hermitian Operators
I. HERMETIAN ADJOINT  35.1

A. If a matrix $A$ acts on a vector space with an inner product structure, then this inner product,

\[ \langle x, y \rangle \quad x, y \in V \]

establish a correspondence between $A$ and its "Hermetian adjoint" by means of the following

Definition (Basis independent def'n)

Let $A: V \to V = \text{complex inner product space}$

Then the Hermetian adjoint $A^H$ of $A$ is defined by

\[ \langle A^H x, y \rangle = \langle x, Ay \rangle \quad \forall x, y \in V \]

This is its defining property at the highest level of abstraction

B. Definition (in the context of matrices)  35.2

A) Let us define the standard complex inner product using the notation

\[ x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad x^T = [x_1 \ldots x_n] \]

\[ x^H = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix} \]

\[ \langle x, y \rangle = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n = x^T y = x^H y \]

B) Apply this inner product to the definition of the Hermetian adjoint of $A$

\[ x^H Ay = \langle x, Ay \rangle = \langle A^H x, y \rangle = (A^H x)^T y \]

\[ = (A^H x)^T y = x^T (A^H)^T y \]

\[ = x^T \overline{(A^H)^T} y \quad \forall x, y \]

C) Conclusion:

\[ A = (A^H)^T \]

or

\[ A^H = A \]
Comment

Definition (Defining a term of matrix elements)

Let $\{u_i\}$ be a basis for $V$ and evaluate

$$\langle A u_i, A u_j \rangle = \langle x_i, A^* y_j \rangle$$

and obtain the matrix elements of $A^*$ relative to the given basis.

1. The matrix elements of $A$ and $A^*$ relative to the given basis.

2. The matrix elements of $A$ and $A^*$ obtained from $A$ by taking the transpose and complex conjugate of the matrix elements of $A$. 

$$A_{ij} = \bar{A_{ji}}$$
II. HERMITIAN OPERATORS

Definition (Hermitian operator)
An operator $A : V \rightarrow V$ is said to be Hermitian with respect to the inner product $\langle \cdot, \cdot \rangle$ if

$A^H = A$

or

$\langle Ax, y \rangle = \langle x, Ay \rangle \quad \forall x, y$

or

$A_{ij} = A_{ji}$

Examples:

(i) $\begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$ is Hermitian on $\mathbb{C}^2$ w.r.t standard inner product

(ii) $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$ is Hermitian on $\mathbb{R}^2$ w.r.t standard inner product

Comment:
If the inner product is real, then $A$ is Hermitian, matrix has real entries, and one has $A^H = A$, i.e. the matrix $A$ is symmetric.

III. CONSTITUTIVE PROPERTIES OF A HERMITIAN MATRIX ("OPERATOR")

Property 1

$A^H = A$ implies the quadratic form $\langle x, Ax \rangle = \bar{x}^T A \bar{x}$ is real. Indeed, one has $\langle x, Ax \rangle = \langle Ax, x \rangle = \langle A^H x, x \rangle = \langle x, Ax \rangle$.

Property 2

$A^H = A$ implies the eigenvalues of $A$ are real. Indeed, one has $Ax = \lambda x \Rightarrow \langle x, Ax \rangle = \langle x, \lambda x \rangle = \lambda \langle x, x \rangle$

$\langle x, Ax \rangle$ a real $\Rightarrow \lambda x$'s real $\Rightarrow \lambda$ is real
Property 3

1) \( A^H = A \) implies eigenvectors of different eigenvalues are orthogonal. Indeed, one has

\[ A x_i = \lambda_i x_i \Rightarrow \langle x_i, A x_i \rangle = \langle x_i, \lambda_i x_i \rangle \]

\[ \langle Ax_i, x_i \rangle - \lambda_i \langle x_i, x_i \rangle \]

\[ = \lambda_i \langle x_i, x_i \rangle \]

Thus \( (\lambda_2 - \lambda_1) \langle x_2, x_1 \rangle = 0 \)

\( \lambda_2 \) is real. Also \( \lambda_2 \neq \lambda_1 \). Thus

\[ \langle x_2, x_2 \rangle = 0 \]

b) The matrix of normalized eigenvectors of \( A \),

\[ U = [x_1 \ x_2 \ \cdots \ x_n] \]

has the property that its columns are orthonormal:

\[ \langle x_i, x_j \rangle = x_i^H x_j = \delta_{ij} \]

Consequently,

\[ U^H U = I \]

\[ (i) \quad U^H = U^{-1} \quad I U U^{-1} = I \]

\[ (ii) \quad \text{Apply row reduction to } U x = I, \]

one finds

\[ x = U^{-1} = U^H \quad U U^{-1} = I \quad (\#) \]

\[ (iii) \quad \text{Multiply Eq.}(\#) \text{ by } U^{-1}, \text{ use } U^{-1} U = I \]

and obtain

\[ U^{-1} U^{-1} = U^{-1} \]

Thus \[ U^H = U^{-1} \]

Definition

A matrix which satisfies \( U^H = U^{-1} \)

is called a unitary matrix.

Property 4

\[ A^H = A \Rightarrow \exists \text{ a unitary matrix } U \text{ s.t.} \]

\[ U^H A U = \Lambda \text{ (diagonal matrix)} \]

and hence

\[ A = U \Lambda U^H \]
Comment 1

1. The diagonalizability of a Hermitian matrix $A$ holds even if the eigenvalue polynomial of $A$ has degenerate roots; in other words, a Hermitian matrix is never defective. This claim will be validated in a subsequent lecture in the context of "normal matrices."

Comment 2

2. The diagonalizability of $A$ implies that its input-output processor representation is mathematized by the matrix equation:

$$A = \lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_n X_n$$

where

$$P_i = P_i^2 = P_i, \quad i = 1, 2, \ldots, n$$

are projection operators that project onto the $i$th eigenspace of $A$, as noted on the previous page. The boxed equation is also known as the spectral representation (or the spectral theorem) for a Hermitian matrix.

Comment 3

3. If one replaces the eigenvalues with the number one, one obtains the relation:

$$I = \sum_{i} x_i x_i^\dagger$$

which is obtained from the eigenvector expansion of $B = \sum x_i <x_i|x_b> = \sum x_i x_i^\dagger b$ because it holds $\forall b$. 
The boxed equation is called the completeness relation for the set of eigenvalues $\lambda_k$.

Thus one has the result that the eigenvectors of $A$ are orthonormal and they form a complete set, i.e., they satisfy the completeness relation

Eq.(*) on p. 35, 10