LECTURE 35

I. Hermetian Adjoint

Definition: Basis Independent
Definition: Matrix Version
Definition: Matrix element-wise

II. Hermetian Operators

III. Properties of Hermetian Operators

I. HERMETIAN ADJOINT

A. If a matrix A acts on a vector space with an inner product structure, then this inner product,

$$\langle x, y \rangle \ , \quad x, y \in V$$

establish a correspondence between A and its "Hermetian adjoint" by means of the following

Definition (Basis independent def'n)

Let $A: V \to V$ = complex inner product space

Then the Hermetian adjoint A^H of A is defined by

$$\boxed{\langle A^H x, y \rangle = \langle x, Ay \rangle \qquad \forall x, y \in V}$$

This is its defining property at the highest level of abstraction

B. Definition (in the context of matrices)

a) Let us define the standard complex inner product using the notation

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad x^T = [x_1 \ \cdots \ x_n]$$
$$x^H = [\bar{x}_1 \ \cdots \ \bar{x}_n]$$

$$\langle x, y \rangle = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n = \overline{x}^T y \equiv x^H y$$

b) Apply this inner product to the definition of the Hermetian adjoint of A

$$x^H A y = \langle x, Ay \rangle = \langle A^H x, y \rangle = (A^H x)^H y$$
$$= \overline{(A^H x)}^T y = x^T \overline{(A^H)}^T y$$
$$= x^H \overline{(A^H)}^T y \qquad \forall x, y$$

c) Conclusion:

$$\boxed{\begin{aligned} A &= \overline{(A^H)}^T \\ \text{or} \quad A^H &= \overline{A^T} \end{aligned}}$$

C.) Definition (Definition in terms of matrix elements) (in the context of a chosen basis)

Let $\{u_i\}$ be a basis for V and evaluate

$$\langle A^H x, y\rangle = \langle x, Ay\rangle = \langle x^i u_i, A\, y^j u_j\rangle$$

$$= \bar{x}^i \langle u_i, A u_j\rangle y^j$$

$$\equiv \bar{x}^i \quad A_{ij} \quad y^j \qquad (\star)$$

$$\langle A^H x, y\rangle = \bar{x}^i \langle A^H u_i, u_j\rangle y^j$$

$$= \bar{x}^i \underbrace{\langle y_j, A^H u_i\rangle}\, y^j$$

$$\equiv \bar{x}^i \quad \overline{A^H_{ji}} \quad y^j \qquad (\star\star)$$

This holds $\forall\, x, y \in V$.
Compare $(\star)$ with $(\star\star)$ and obtain

$$\boxed{A_{ij} = \overline{A^H_{ji}}}$$

or

$$A^H_{ij} = \overline{A_{ji}} = \overline{A^T_{ij}}$$

Comments

1. A_{ij} are the matrix elements of A relative to the given basis
and
A^H_{ij} are the matrix elements of A^H relative to the given basis.

2. The matrix elements of A^H are obtained from A by taking the transpose and complex conjugate of the matrix elements of A.

IV. HERMETIAN OPERATORS

Definition (Hermetian operator)

An operator $A : V \to V$ is said to be **Hermetian** with respect to the inner product $\langle\, ,\, \rangle$ if

$$\boxed{\begin{array}{l} A^H = A \\ \text{or} \quad \langle Ax, y\rangle = \langle x, Ay\rangle \quad \forall x, y \\ \text{or} \quad A_{ij} = \overline{A_{ji}} \end{array}}$$

Examples:

(i) $\begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$ is Hermetian on C^2 w.r.t. standard inner product

(ii) $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$ is Hermetian on R^2 w.r.t. standard inner product

Comment.

If the inner product is real, then a Hermetian matrix has real entries, and one has

$$A^T = A,$$

i.e the matrix A is symmetric.

III. CONSTITUTIVE PROPERTIES OF A HERMETIAN MATRIX (="OPERATOR")

Property 1

$A^H = A$ implies the quadratic form

$$\langle x, Ax\rangle = \overline{x^i} A_{ij} x^j$$

is real. Indeed, one has

$$\overline{\langle x, Ax\rangle} = \langle Ax, x\rangle = \langle A^H x, x\rangle = \langle x, Ax\rangle$$

Property 2

$A^H = A$ implies the eigenvalues of A are real. Indeed, one has

$$Ax = \lambda x \Rightarrow \langle x, Ax\rangle = \langle x, \lambda x\rangle = \lambda \|x\|^2$$

$$\langle x, Ax\rangle \text{ is real} \Rightarrow \lambda\|x\|^2 \text{ is real} \Rightarrow \lambda \text{ is real}$$

Property 3

a) $A^H = A$ implies eigenvectors of different eigenvalues are orthogonal. Indeed on has

$$A x_1 = \lambda_1 x_1 \Rightarrow \underbrace{\langle x_2, A x_1 \rangle}_{\underbrace{\langle A x_2, x_1 \rangle}_{\bar{\lambda}_2 \langle x_2, x_1 \rangle}} = \underbrace{\langle x_2, \lambda_1 x_1 \rangle}_{\lambda_1 \langle x_2, x_1 \rangle}$$

Thus $(\bar{\lambda}_2 - \lambda_1)\langle x_2, x_1 \rangle = 0$

λ_2 is real. Also $\lambda_2 \neq \lambda_1$. Thus

$$\boxed{\langle x_2, x_1 \rangle = 0}$$

b) The matrix of normalized eigenvectors of A,

$$U = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}$$

has the property that its columns are orthonormal

$$\langle x_i, x_j \rangle = x_i^H x_j = \delta_{ij}.$$

Consequently,

$$U^H U = I$$

∴

(i) $U^H = U_L^{-1}$: $U_L^{-1} U = I$

(ii) Apply row reduction to $UX = I$, one finds

$$X = U_R^{-1} : \quad U U_R^{-1} = I \qquad (*)$$

(iii) Multiply Eq. $(*)$ by U_L^{-1}, use $U_L^{-1} U = I$ and obtain

$$U_R^{-1} = U_L^{-1} \equiv U^{-1}$$

Thus $\boxed{U^H = U^{-1}}$

Definition

A matrix which satisfies

$$U^H = U^{-1}$$

is called a unitary matrix.

Property 4

$A^H = A \Rightarrow \exists$ a unitary matrix U s.t.

$$U^H A U = \Lambda \quad (\text{diagonal matrix})$$

and hence

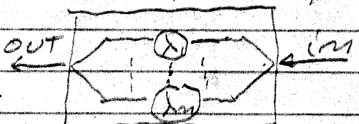
$$A = U \Lambda U^H$$

Comment 1

1. The diagonalizability of a Hermetian matrix A holds even if the eigenvalue polynomial of A has degenerate roots; in other words, a Hermetian matrix is never defective. This claim will be validated in a subsequent lecture in the context of "normal matrices".

Comment 2

2. The diagonalizability of A implies that its input-output processor representation



is mathematized by the matrix eq'n

$$\boxed{\begin{aligned} A &= X_1\lambda_1X_1^H + X_2\lambda_2X_2^H + \cdots + X_n\lambda_nX_n^H \\ &= \lambda_1P_1 + \lambda_2P_2 + \cdots + \lambda_nP_n \text{; with } \lambda_i = \underline{\text{real}} \end{aligned}}$$

where

$$P_i : P_i^2 = P_i \qquad i = 1, 2, \cdots, n$$

are projection operators that project onto the i^{th} eigenspace of A.

The boxed equation (on the previous page) is also known as the spectral representation (or the spectral theorem) for a Hermetian matrix

Comment 3

3. If one replaces the eigenvalues with the number one, one obtains the relation

$$\boxed{I = \sum_i X_i X_i^T}$$ "completeness (*) relation"

which is obtained from the eigenvector expansion of

$$\vec{b} = \sum \vec{X}_i \langle X_i | b \rangle = \sum_i X_i X_i^T b$$

because it holds $\forall b$.

The boxed equation is called the <u>completeness relation</u> for the set of eigenvectors $\{\vec{X}_i\}$.

Thus one has the result that

<u>the eigenvectors of A are orthonormal</u>

<u>and</u> <u>they form a complete set,</u>

<u>i.e. they satisfy the completeness relation,</u>

Eq. (*) on p 35.10