LECTURE 35

I. Hermetian Adjoint (Ch5 in Strang's book)

Definition: Basis Independent
Definition: Matrix Version
Definition: Matrix element-wise

II. Hermetian Operators

III. Properties of Hermetian operators

I. HERMETIAN ADJOINT

We shall now mathematize the concept "energy" and "intensity" as it arises in linear (e.g. electromagnetic, acoustic, oscillatory, ...) systems.

This is done by focussing on the quadratic nature of these concepts and expressing them in terms of inner products.

The starting point of the mathematization process is a complex vector space with an inner product structure,

$$\langle x, y \rangle\,, \quad x, y \in V$$

A concrete example of this is $\mathbb{C}^n$ with the standard complex inner product:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}; \; y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad \begin{matrix} x^T = [x_1, \dots, x_n] \\ x^H = [\bar{x}_1, \dots, \bar{x}_n] \end{matrix}$$

$$\langle x, y \rangle = \bar{x}_1 y_1 + \dots + \bar{x}_n y_n = \sum_i \bar{x}_i y_i \equiv \bar{x}^T y \equiv x^H y$$

A) Let us now describe the process by which one arrives at a new concept, namely A^H, the <u>Hermetian adjoint</u> of a matrix

Given: (i) Matrix

$$A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$
$$y \rightsquigarrow Ay = z$$

and

(ii) an inner product, such as the one above one can identify a new matrix A^H as follows:

CONT'D on p 35, 2b

CONTINUED from P 35, 2a

$$\langle x, Ay\rangle = \sum_i \bar{x}_i (Ay)_i \quad (= x^T A y)$$

$$= \sum_i \bar{x}_i \sum_j A_{ij} y_j$$

$$= \sum_i \sum_j \bar{x}_i A^T_{ji} y_j$$

$$= \sum_i \sum_j \overline{\bar{A}^T_{ji} x_i} \; y_j$$

Let $\bar{A}^T_{ji} = A^H_{ji}$, interchange $\sum_i$ and $\sum_j$. Then

$$= \sum_j \overline{\left(\sum_i A^H_{ji} x_i\right)} \; y_j$$

$$= \sum \overline{(A^H x)_j} \; y_j$$

$$\langle x, Ay\rangle = \langle A^H x, y\rangle \qquad \forall\, x, y$$

Conclusion: This process defines.

$\boxed{A^H = \bar{A}^T}$; $\left(A = \overline{(A^H)^T}\right)$

GO TO page 35,2 c.

B) Abstract(*) Definition of A^H.

Let $A: V \to V$ = complex inner product space

Then the Hermetian adjoint A^H of A is defined by the condition

$$\boxed{\langle A^H x, y\rangle = \langle x, Ay\rangle \quad \forall\, x, y \in V}$$

(Matrix notation: $(A^H x)^H y = x^+ A y$)

This is its defining (basis independent, i.e. subsuming any choice of basis) property at the highest level of abstraction.

Go To p35, 3

(*) Here "Abstract" means more general, hierarchically at a higher level, encompasing a larger number of special cases, hence cognitively more powerful.

CONT'D from p35.2C.

C.) Definition (Definition in terms of matrix elements) (in the context of a chosen basis)

Let $\{u_i\}$ be a basis for V and evaluate

$$\langle A^H x, y\rangle = \langle x, Ay\rangle = \langle x^i u_i, A\, y^j u_j\rangle$$

$$= \bar{x}^i \underbrace{\langle u_i, A u_j\rangle}\, y^j$$

$$\equiv \bar{x}^i \quad A_{ij} \quad y^j \qquad (\star)$$

$$\langle A^H x, y\rangle = \bar{x}^i \langle A^H u_i, u_j\rangle\, y^j$$

NOT $\overline{\langle y_j, A^H u_i\rangle}$ $\quad = \bar{x}^i\, \underbrace{\overline{\langle u_j, A^H u_i\rangle}}\, y^j$

$$\equiv \bar{x}^i \quad \overline{A^H_{ji}} \quad y^j \qquad (\star\star)$$

This holds $\forall\, x, y \in V$.
Compare $(\star)$ with $(\star\star)$ and obtain

$$\boxed{A_{ij} = \overline{A^H_{ji}}}$$

or $\boxed{\overline{A^H_{ij}} = \overline{A_{ji}} = \overline{A^T_{ij}}}$

Comments

1. A_{ij} are the $\boxed{\text{matrix elements}}$ of A <u>relative to the given basis</u>

and

A^H_{ij} are the $\boxed{\text{matrix elements}}$ of A^H relative to the given basis.

2. The matrix elements of A^H are obtained from A by taking the <u>transpose</u> and <u>complex conjugate</u> of the matrix elements of A.

IV. HERMETIAN OPERATORS

Definition (Hermetian operator)

An operator $A : V \to V$ is said to be <u>Hermetian</u> with respect to the inner product $\langle \, , \, \rangle$ if

$$\boxed{\begin{array}{l} A^H = A \\ \text{or} \quad \langle Ax, y \rangle = \langle x, Ay \rangle \quad \forall x, y \\ \text{or} \quad A_{ij} = \overline{A_{ji}} \end{array}}$$

Examples:

(i) $\begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$ is Hermetian on C^2 w.r.t. standard inner product

(ii) $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$ is Hermetian on R^2 w.r.t. standard inner product

Comment.

If the inner product is real, then a Hermetian matrix has real entries, and one has

$$A^T = A,$$

i.e the matrix A is symmetric.

III. CONSTITUTIVE PROPERTIES OF A HERMETIAN MATRIX (="OPERATOR")

Property 1

$A^H = A$ implies the quadratic form

$$\langle x, Ax\rangle = \bar{x}^i A_{ij} x^j$$

is real. Indeed, one has

$$\overline{\langle x, Ax\rangle} = \langle Ax, x\rangle = \langle A^H x, x\rangle = \langle x, Ax\rangle$$

Property 2

$A^H = A$ implies the eigenvalues of A are real. Indeed, one has

$$Ax = \lambda x \Rightarrow \langle x, Ax\rangle = \langle x, \lambda x\rangle = \lambda \|x\|^2$$

$$\langle x, Ax\rangle \text{ is real} \Rightarrow \lambda \|x\|^2 \text{ is real} \Rightarrow \lambda \text{ is real}$$

Property 3

a) $A^H = A$ implies eigenvectors of different eigenvalues are orthogonal. Indeed on has

$$A x_1 = \lambda_1 x_1 \Rightarrow \underbrace{\langle x_2, A x_1\rangle}_{\underbrace{\langle A x_2, x_1\rangle}_{\bar\lambda_2 \langle x_2, x_1\rangle}} = \underbrace{\langle x_2, \lambda_1 x_1\rangle}_{\lambda_1 \langle x_2, x_1\rangle}$$

$A x_2 = \lambda_2 x_2$

$\langle x_1, A x_2\rangle = \lambda_2 \langle x_1, x_2\rangle$

$\langle A x_2, x_1\rangle = \bar\lambda_2 \langle x_2, x_1\rangle$

$\langle x_2, A x_1\rangle = \bar\lambda_2 \langle x_2, x_1\rangle$

Thus $(\bar\lambda_2 - \lambda_1)\langle x_2, x_1\rangle = 0$

λ_2 is real. Also $\lambda_2 \neq \lambda_1$. Thus

$$\boxed{\langle x_2, x_1\rangle = 0}$$

b) The matrix of <u>normalized</u> eigenvectors of A,

$$U = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}$$

has the property that its columns are orthonormal

$$\langle x_i, x_j\rangle = x_i^H x_j = \delta_{ij}.$$

Consequently,

$$\left.\begin{array}{l} U^H U = I \\ U_L^{-1} U = I \end{array}\right\} \therefore U_L^{-1} = U^H$$

(ii) Apply row reduction to $UX = I$, one finds the unique matrix:

$$X = U_R^{-1} : \quad U U_R^{-1} = I \qquad (*)$$

(iii) Multiply Eq. (*) by U_L^{-1}, use $U_L^{-1} U = I$ and obtain

$$U_R^{-1} = U_L^{-1} \equiv U^{-1}$$

Thus $\boxed{U^H = U^{-1}}$

<u>Definition</u>

A matrix which satisfies

$$U^H = U^{-1}$$

is called a <u>unitary matrix</u>.

<u>Property 4</u>

$A^H = A \Rightarrow \exists$ a unitary matrix U s.t.

$$U^H A U = \Lambda \text{ (diagonal matrix)}$$

and hence

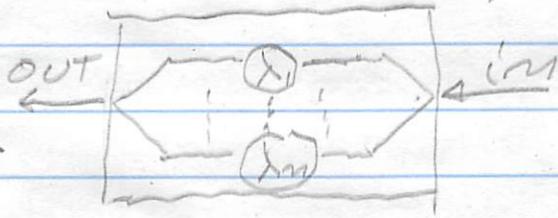
$$A = U \Lambda U^H$$

Comment 1.

1. The diagonalizability of a Hermetian matrix A holds even if the eigenvalue polynomial of A has degenerate roots; in other words, a Hermetian matrix is never defective. Thus claim will be validated in a subsequent lecture in the context of "normal matrices".

Comment 2

2. The diagonalizability of A implies that its input-output processor representation



is mathematized by the matrix eq'n

$$A = U \Lambda U^H = \begin{bmatrix} | \\ X_1 \\ | \end{bmatrix} \lambda_1 \begin{bmatrix} - X_1 - \end{bmatrix}^H + \begin{bmatrix} | \\ X_2 \\ | \end{bmatrix} \lambda_2 \begin{bmatrix} - X_2 - \end{bmatrix}^H + \cdots + \begin{bmatrix} | \\ X_n \\ | \end{bmatrix} \lambda_n \begin{bmatrix} - X_n - \end{bmatrix}^H$$

$$= \lambda_1 \begin{bmatrix} | \\ X_1 \\ | \end{bmatrix} \begin{bmatrix} - X_1 - \end{bmatrix}^H + \lambda_2 \begin{bmatrix} | \\ X_2 \\ | \end{bmatrix} \begin{bmatrix} - X_2 - \end{bmatrix}^H \cdots + \lambda_n [\ ][\ ] \quad \text{with } \lambda_i = \underline{\text{real}}$$

$$= \lambda_1 P_1 + \lambda_2 P_2 \cdots + \lambda_n P_n$$

(Spectral rep'n)

where

$P_i : P_i^2 = P_i \qquad i = 1, 2, \cdots, n$

are <u>projection operators</u> that project onto the i^{th} eigenspace of A.

The boxed equation on the previous page is also known as the <u>spectral representation</u> (or the <u>spectral theorem</u>) for a Hermetian matrix

<u>Comment 3</u>

3. If one replaces the eigenvalues with the number one, one obtains the relation

$$\boxed{I = \sum_i X_i X_i^\dagger} \quad (*) \text{ "completeness relation"}$$

which is obtained from the eigenvector expansion of

$$\vec{b} = \sum \vec{X}_i \langle X_i | b \rangle = \sum_i X_i X_i^\dagger b$$

because it holds $\forall b$.

The boxed equation is called the <u>completeness relation</u> for the set of eigenvectors $\{\vec{X}_i\}$.

Thus one has the result that the eigenvectors of A are orthonormal and they form a complete set, i.e. they satisfy the completeness relation, Eq. $(*)$ on p 35.10