LECTURE 35

I. Hermetian Adjoint (Ch 5 in Strang's book)
   Definition: Basis Independent
   Definition: Matrix Version
   Definition: Matrix element-wise

II. Hermetian Operators

III. Properties of Hermetian Operators
I. HERMETIAN ADJOINT

We shall now mathematize the concept "energy" and "intensity" as it arises in linear (e.g., electromagnetic, acoustic, oscillatory...) systems.

This is done by focusing on the quadratic nature of these concepts and expressing them in terms of inner products.

The starting point of the mathematization process is a complex vector space with an inner-product structure,

\[ \langle x, y \rangle, \quad x, y \in V \]
A concrete example of this is \( \mathbb{C}^n \) with the standard complex inner product:

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix},
\begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_n
\end{bmatrix}
\]

\[
X^T = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix},
X^H = \begin{bmatrix} \overline{x}_1 & \cdots & \overline{x}_n \end{bmatrix}
\]

\[
\langle x, y \rangle = \overline{x}_1 y_1 + \cdots + \overline{x}_n y_n = \sum_{i=1}^n \overline{x}_i y_i = \overline{x}^T y = x^H y
\]

a) Let us now describe the process by which one arrives at a new concept, namely \( A^* \), the Hermitian adjoint of a matrix.

Given: (i) Matrix

\[
A : \mathbb{C}^n \to \mathbb{C}^n
\]

\[
y \mapsto Ay = z
\]

and

(ii) an inner product, such as the one above one can identify a new matrix \( A^* \) as follows:

CONT'D on p 35.26
CONTINUED from P 35.2a

\[ \langle x, Ay \rangle = \sum_{i} x_{i} (Ay)_{i} = x^{T} Ay \]

\[ = \sum_{i} x_{i} \sum_{j} A_{i}^{j} y_{j} \]

\[ = \sum_{i} \sum_{j} x_{i} A_{j}^{i} y_{j} \]

\[ = \sum_{i} \sum_{j} \overline{A_{j}^{i}} x_{i} y_{j} \]

Let \[ \overline{A_{j}^{i}} = A_{i}^{j} \], interchange \( \sum_{i} \) and \( \sum_{j} \). Then

\[ = \sum_{j} \left( \sum_{i} A_{i}^{j} x_{i} \right) y_{j} \]

\[ = \sum_{j} (A^{H} x)_{j} y_{j} \]

\[ \langle x, Ay \rangle = \langle A^{H} x, y \rangle \quad \forall x, y \]

Conclusion: This process defines.

\[ A^{H} = A^{T} \quad (A = (A^{H})^{T}) \]

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B) "Abstract Definition of A''

Let \( A : V \rightarrow V \) = complex inner product space

Then the Hermitian adjoint \( A'' \) of \( A \)

is defined by the condition

\[ \langle A''x, y \rangle = \langle x, Ay \rangle \quad \forall x, y \in V \]

(Matrix notation: \( (A''x)^\ast y = x^\ast Ay \))

This is its defining (basis independent, i.e. subsuming any choice of basis) property at the highest level of abstraction.

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(*) Here "Abstract" means more general, hierarchically at a higher level, encompassing a larger number of special cases, hence cognitively more powerful.
C) Definition (Definition in terms of matrix elements)

Let \( \{ \mathbf{u}_i \} \) be a basis for \( V \) and evaluate

\[
\langle A^H x, y \rangle = \langle x, A y \rangle = \langle x^2 \mathbf{u}_i, A y^2 \mathbf{u}_j \rangle
\]

\[
= \overline{x}^2 \langle \mathbf{u}_i, A \mathbf{u}_j \rangle y^2
\]

\[
= \overline{x}^2 A_{ij} y^2 \quad \text{(**)}
\]

Let \( \langle y, A^H u_i \rangle = \overline{x}^2 \langle u_i, A^H u_i \rangle y^2 \)

\[
\overline{\langle y, A^H u_i \rangle} = \overline{x}^2 \langle u_i, A^H u_i \rangle y^2
\]

\[
= \overline{x}^2 A_{ji} y^2 \quad \text{(***)}
\]

This holds \( \forall x, y \in V \).

Compare (***) with (**) and obtain

\[
A_{ij} = A_{ji}^H
\]

or

\[
A_{ij}^H = A_{ji} = A_{ij}^T
\]
Comments

1. $A_{ij}$ are the **matrix elements** of $A$ relative to the given basis.

   and

   $A^H_{ij}$ are the **matrix elements** of $A^H$ relative to the given basis.

2. The matrix elements of $A^H$ are obtained from $A$ by taking the transpose and complex conjugate of the matrix elements of $A$. 
II. HERMETIAN OPERATORS

Definition (Hermetian operator)

An operator $A : V \to V$ is said to be Hermetian with respect to the inner product $\langle \cdot, \cdot \rangle$ if

$$A^H = A$$

or

$$\langle Ax, y \rangle = \langle x, Ay \rangle \quad \forall x, y$$

$$A_{ij} = A_{ji}$$

Examples:

1. $\begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$ is Hermetian on $\mathbb{C}^2$ w.r.t. standard inner product.

2. $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$ is Hermetian on $\mathbb{R}^2$ w.r.t. standard inner product.
Comment
If the inner product is real, then a Hermitian matrix has real entries, and one has
\[ A^H = A \]
\[ \text{i.e. the matrix } A \text{ is symmetric.} \]

III. CONSTITUTIVE PROPERTIES OF A HERMITIAN MATRIX (= "OPERATOR")

Property 1
\[ A^H = A \text{ implies the quadratic form} \]
\[ \langle x, Ax \rangle = \bar{x}^i A_{i,j} x^j \]
\[ \text{is real. Indeed, one has} \]
\[ \langle x, Ax \rangle = \langle Ax, x \rangle = \langle A^H x, x \rangle = \langle x, A^H x \rangle \]

Property 2
\[ A^H = A \text{ implies the eigenvalues of } A \text{ are real. Indeed, one has} \]
\[ Ax = \lambda x \Rightarrow \langle x, Ax \rangle = \langle x, \lambda x \rangle = \lambda \|x\|^2 \]
\[ \langle x, Ax \rangle \text{ is real } \Rightarrow \|x\|^2 \text{ is real } \Rightarrow \lambda \text{ is real} \]
Property 3

a) $A^H = A$ implies eigenvectors of different eigenvalues are orthogonal. Indeed has

$$A^* x_1 = \lambda_1 x_1 \Rightarrow \langle x_2, A x_1 \rangle = \langle x_2, \lambda_1 x_1 \rangle$$

Thus $(\lambda_2 - \lambda_1) \langle x_2, x_1 \rangle = 0$

$\lambda_2$ is real. Also $\lambda_2 \neq \lambda_1$. Thus

$$\langle x_2, x_2 \rangle = 0$$

b) The matrix of normalized eigenvectors of $A$,

$$U = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}$$

has the property that its columns are orthonormal

$$\langle x_i^*, x_j \rangle = x_i^H x_j = \delta_{i,j}$$

consequently
\[ U^H U = I \]
\[ U^{-1} U = I \] \[ \therefore U^{-1} = U^H \]

(ii) Apply row reduction to \( UX = I \), one finds the unique matrix:
\[ X = U_L^{-1} U_R \Rightarrow U_U^{-1} U_R = I \] (4)

(iii) Multiply Eq. (4) by \( U_L^{-1} \), use \( U_L^{-1} U = I \) and obtain:
\[ U_R^{-1} = U_L^{-1} \equiv U^{-1} \]

Thus, \[ U^H = U^{-1} \]

**Definition**
A matrix which satisfies \[ U^H = U^{-1} \]
is called a unitary matrix.

**Property 4**
\[ A^H = A \quad \Rightarrow \quad \exists \text{ a unitary matrix } U \text{ s.t.} \]
\[ U^H A U = \Lambda \ (\text{diagonal matrix}) \]
and hence
\[ A = U \Lambda U^H \]
1. The diagonalizability of a Hermetian matrix $A$ holds even if the eigenvalue polynomial of $A$ has degenerate roots; in other words, a Hermetian matrix is never defective. This claim will be validated in a subsequent lecture in the context of "normal matrices."

Comment 2

2. The diagonalizability of $A$ implies that its input-output processor representation is mathematized by the matrix eq'n

$$A = U \Lambda U^H = \begin{bmatrix} \lambda_1 [x_1] - x_1^H \\ \lambda_2 [x_2] - x_2^H \\ \vdots \\ \lambda_m [x_m] - x_m^H \end{bmatrix}$$

$$= \lambda_1 [x_1] - x_1^H + \lambda_2 [x_2] - x_2^H + \cdots + \lambda_m [x_m] - x_m^H$$

With $\lambda_i = \text{real}$.

$$= \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m \quad \text{(Spectral rep'n)}$$
where
\[ P_i : P_i^2 = P_i \quad i = 1, 2, \ldots, n \]
are projection operators that project onto the \( i \)-th eigenspace of \( A \).
The boxed equation is also known as the spectral representation (or the spectral theorem) for a Hermitian matrix.

Comment 3

3. If one replaces the eigenvalues with the number one, one obtains the relation
\[ I = \sum_{i} X_i X_i^T \]
"completeness relation" which is obtained from the eigenvector expansion of
\[ b = \sum X_i \langle x_i | b \rangle = \sum x_i x_i^T b \]
because it holds \( A b \).
The boxed equation is called the completeness relation for the set of eigenvectors $\exists x_i$. 

Thus one has the result that the eigenvectors of $A$ are orthonormal and they form a complete set, i.e. they satisfy the completeness relation $\text{Eq.}(\ast)$ on p 35.11.