LECTURE 36

I. SKEW HERMETIAN MATRICES
and their properties
a) Their eigenvalues
b) Their eigenvectors

II UNITARY MATRICES
and their properties
a) Their eigenvalues
b) Their eigenvectors.

III. UNITARY DYNAMICAL SYSTEMS

IV. SIMILARITY TRANSFORMATIONS
(revisited)

I. SKEW SYMMETRIC MATRICES

Consider a Hermetian matrix such as

$$A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} ; \quad A^H = A$$

Then

$$K = iA = \begin{bmatrix} 2i & 3i+3 \\ 3i-3 & 5i \end{bmatrix} ; \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^H = -\begin{bmatrix} & \end{bmatrix}$$

has the property

$$K^H = (iA)^H = i^H A^H = (-i)A = -K$$

Thus quite generally one has the following

Definition

A matrix (or a linear operator) K is said to be skew-Hermetian if

$$K^H = -K.$$

One also has the following

Proposition

If K is skew-Hermetian then $A = iK$ is Hermetian ; $A^H = (i)^H K^H = (-i)(-K) = iK$

If A is Hermetian, then $K = iA$ is skew-Hermetian

$$K^H = (iA)^H = -iA = -K$$

The four constitutive properties (pages 35.6 – 35.8) of Hermetian matrices imply a corresponding set of constitutive properties for skew-Hermetian matrices.

Properties

(1') For all x, $x^H K x$ is purely imaginary

(2') Every eigenvalue of K is imaginary

(3') The eigenvectors of different eigenvalues are orthogonal

(4') $\exists$ a unitary U such that

$$U^{-1} K U = \Lambda$$
$$K = U \Lambda U^{-1}$$

where Λ is a purely imaginary matrix.

II. UNITARY MATRICES

Rotations on a real vector space preserve the length of vectors and the angles between them; in other words, rotations preserve the shape and size of bodies in an inner product space

Unitary tranformations extend the concept of rotation from real to complex inner product spaces

The defining property of unitary transformation U is

$$\boxed{UU^H = U^H U = I} \text{ i.e. } \boxed{U^H = U^{-1}}$$

A unitary transformation numerous constitutive properties

Property 1

(1) U preserves the norm (= squared length) of a vector

$$\|Ux\|^2 = \langle Ux, Ux\rangle = \|x\|^2 \qquad \forall x$$

Comment:

$\langle Ux, Ux\rangle = \langle x, x\rangle\ \forall x$ implies $\langle Ux, Uy\rangle = \langle x, y\rangle\ \forall x,y$

The validation of this fact is a good exercise.

Property 2

The eigenvalues of U are of unit modulus.

$$Ux = \lambda x \Rightarrow |\lambda|^2 = 1, \text{ i.e. } \lambda = e^{i\alpha}$$

Property 3

The eigenvectors of U corresponding to different eigenvalues are orthogonal

$$\left.\begin{array}{l} Ux = \lambda x \\ Uy = \sigma y \end{array}\right\} \Rightarrow \langle x, y\rangle = \langle Ux, Uy\rangle = \bar{\lambda}\sigma\langle x, y\rangle = e^{i(\beta-\alpha)}\langle x, y\rangle$$

$$\underbrace{\left[1 - e^{i(\beta-\alpha)}\right]}_{\neq \text{ zero}}\langle x, y\rangle = 0 \Rightarrow \langle x, y\rangle = 0$$

Property 4

U is diagonalized by a unitary similarity transformation, i.e. $\exists S$ s.t.

$$\boxed{\begin{array}{l} S^H U S = \Lambda = \begin{bmatrix} e^{i\alpha_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\alpha_n} \end{bmatrix} \\ \\ U = S \Lambda S^H \end{array}}$$

Discussion

Consider the O.N. eigenvector basis $\{x_i\}$ of U.

determined by U. One has

$$U\underbrace{\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}}_{S} = \underbrace{\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}}_{S} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = \Lambda$$

unitary

Consequently,

$$S^H U S = \Lambda$$
$$U = S \Lambda S^H$$

<u>Property 5</u>

If K is skew-Hermetian, then $e^{Kt} = U$ is unitary.

Comment

K is called the <u>generator</u> of U.

III. Application: UNITARY DYNAMICAL SYSTEMS

Consider a time-invariant linear system with a skew-Hermetian

$$\frac{du}{dt} = Ku$$

Question: How does the squared norm $\|u\|^2 = u^H u$ of the state u change as a function of time?

Answer:

$$\frac{d}{dt}\|u\|^2 = \frac{d}{dt} u^H u = \frac{du^H}{dt} u + u^H \frac{du}{dt}$$
$$= (Ku)^H u + u^H K u$$
$$= \langle Ku, u\rangle + \langle u, Ku\rangle$$
$$= \langle u, K^H u\rangle + \langle u, Ku\rangle$$
$$= -\langle u, Ku\rangle + \langle u, Ku\rangle = 0$$

Conclusion:

For such a system

$$\|u(t)\|^2 = \text{const}$$

i.e. the magnitude of the state is an <u>integral of motion</u>: The evolution of $u(t)$ such that it is <u>independent of time</u>,

$$\|u(t)\|^2 = \|u(0)\|^2 \quad (*)$$

Q: WHAT DO A, K, and U HAVE IN COMMON?

Eigenvalues of $A=A^H$, $K=-K^H$, $U: U^H=U^{-1}$ in the complex λ-plane.

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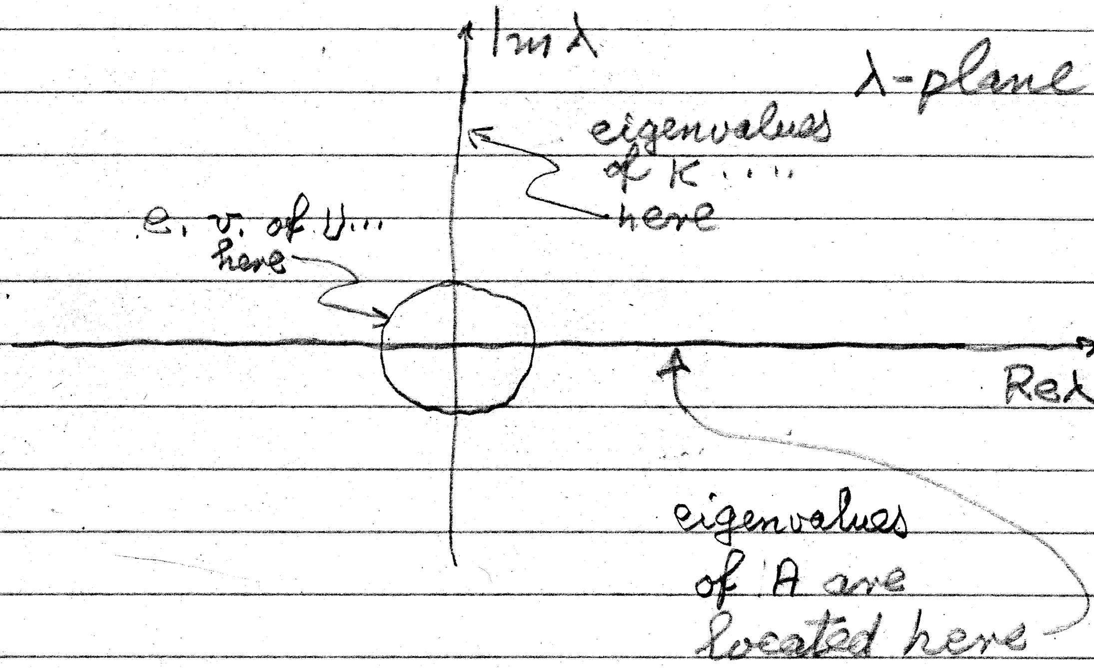

ANS.: $\boxed{A, K, U \text{ are diagonalized by unitary xform's.}}$

Q: What other matrices have this ppty

A: Normal matrices N:

$$NN^H = N^H N$$

Comment

We also know that

$$u(t) = e^{Kt} u(0).$$

Consequently

$$\|u(t)\|^2 = \langle e^{Kt} u(0), e^{Kt} u(0) \rangle$$

$$= \langle (e^{Kt})^H e^{Kt} u(0), u(0) \rangle$$

$$= \langle e^{-Kt} e^{Kt} u(0), u(0) \rangle$$

If, as is shown in the Appendix, that

$$e^{-Kt} e^{Kt} = e^{-(K+K)t} = I,$$

one has the result that

$$\|u(t)\|^2 = \|u(0)\|^2.$$

This is the same result as Eq. (*) on the previous page: A unitary evolution, $U = e^{Kt}$, is norm preserving.

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IV SIMILARITY TRANSFORMATIONS

In Lecture 34 we have highlighted the importance of discovering the characteristic behavior of a time invt linear system

$$\frac{d\vec{u}}{dt} = A\vec{u}$$

If (i) A was diagonalizable and (ii) we were able to find the eigenvector matrix S such that

$$A = S \Lambda S^{-1},$$

we would introduce the eigenbasis representation of $\vec{u}$ by

$$\vec{u}(t) = S\, \vec{v}(t).$$

This change of dependent variables gave us

$$\frac{d\vec{v}(t)}{dt} = \Lambda \vec{v}(t),$$

a completely decoupled system of equations Eq. (**) on page 34.2, for each of the individual independent degrees of freedom,

$$v_i(t) = c_i e^{\lambda_i t} \qquad i = 1, \ldots, n.$$

However, even though such a decoupling may be possible, such an ambitious undertaking may not be necessary. In fact, for a defective A it is not even possible. (However, generalized eigenvectors still provide a solution to the problem. See Lecture 34) Instead, a similarity transformation from A to

$$B = M^{-1}AM$$

might very well suffice. This is because

$$\frac{d\vec{y}(t)}{dt} = B\vec{y}(t), \quad \vec{y} = M^{-1}u(t)$$

would be, for example, a partially decoupled systems whenever B is upper triangular

$$B = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \lambda_n \end{bmatrix}$$

In fact, we shall show (in Lecture 37) that such similarity transformation always exists.

The physical and mathematical significance of a similarity transformation is that the dynamical systems

$$\frac{du}{dt} = Au \quad \text{and} \quad \frac{dy}{dt} = By$$

are mathematically equivalent because the eigenvalues of A and B are the same. This fact is guaranteed by the

<u>Theorem</u>

Similar matrices have the same eigenvalues, i.e

$$A \quad \text{and} \quad B = M^{-1}AM$$

have the same eigenvalues.

1st proof

$$AX = \lambda X$$

$$MBM^{-1}x = \lambda x$$

$$B(M^{-1}x) = \lambda(M^{-1}x) \qquad \text{QED.}$$

2nd proof

$$\det|A - \lambda I| = \det|M(B - \lambda I)M^{-1}| = \det|B - \lambda I|$$

i.e. the characteristic polynomial of B is the same as that of A. Consequently the roots are the same also.