APPENDIX (For Lecture 36)

Proof of $e^a e^b = e^{a+b}$ whenever

(i) e^a and e^b are convergent matrix power series and

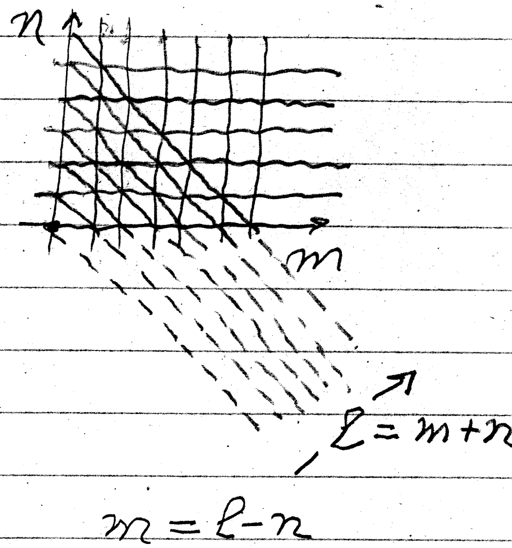
and

(ii) $ab = ba$.

Consider $e^a = \sum_{m=0}^{\infty} \frac{a^m}{m!}$, a convergent series

$e^b = \sum_{n}^{\infty} \frac{b^n}{n!}$ a convergent series

Their product is



$$e^a e^b = \sum_m \frac{a^m}{m!} \sum_n \frac{b^n}{n!}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a^m}{m!} \frac{b^n}{n!}$$

$$= \sum_{\ell=0}^{\infty} \sum_{n=0}^{\ell} \frac{a^{\ell-n}}{(\ell-n)!} \frac{b^n}{n!}$$

Instead of doing the infinite vertical sum $\sum_{n=0}^{\infty}$ for $m = 0, 1, 2, \ldots$

we have done the finite diagonal sum

$\sum_{n=0}^{n=\ell}$ for $\ell = 0, 1, 2, \ldots$.

Thus

$$e^a e^b = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{n=0}^{\ell} \frac{\ell!}{(\ell-n)!\, n!} a^{\ell-n} b^n$$

$$\text{"} = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (a+b)^\ell \quad \text{because } ab = ba$$

$$e^a e^b = e^{a+b} \qquad \text{Q.E.D.}$$