LECTURE 37

I. TRIANGULAR FORM VIA UNITARY MATRIX

A.) Partially Decoupled Time Invariant Linear System.

B.) General Time Invariant Linear System

C.) The Triangularization Theorem

D.) The Triangularization Theorem; Applications

E.) The Triangularization Theorem; Its Proof.

I. TRIANGULAR FORM VIA UNITARY MATRIX

A) Partially Decoupled Time Invariant Linear System.

Consider a time-invariant linear system generated by an upper triangular matrix T:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \overbrace{\begin{bmatrix} \lambda_1 & \alpha & \beta \\ 0 & \lambda_2 & \gamma \\ 0 & 0 & \lambda_3 \end{bmatrix}}^{T} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \equiv T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (*)$$

An example would be based on the uranium decay chain

$$^{238}U + 1\,\text{neutron} \rightarrow \underbrace{{}^{239}U}_{z(t)} \; \underbrace{\xrightarrow[\substack{23.45 \\ \text{minutes}}]{}}_{\frac{-1}{\lambda_3}=\gamma} \; \underbrace{{}^{239}Ne}_{y(t)} \; \underbrace{\xrightarrow[\substack{2.356 \\ \text{days}}]{}}_{\frac{-1}{\lambda_2}=\alpha} \; \underbrace{{}^{239}Pu}_{x(t)} \; \underbrace{\xrightarrow[\substack{24100 \\ \text{years}}]{}}_{\frac{-1}{\lambda_1}\ \boxed{\beta=0}} \; {}^{235}U$$

by which one produces Plutonium in a nuclear reactor via

$$^{238}U + 1\text{ neutron} \rightarrow {}^{239}U$$

A differential equation like Eq. (*) on page 37.1 governs the state

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

of the system whose components make up a mixture of three different kinds of nuclides.

The differential equation is readily solved by the method of back substitution. One has

$$\frac{dx}{dt} - \lambda_1 x = \alpha y + \beta z \qquad (1)$$

$$\frac{dy}{dt} - \lambda_2 y = \gamma z \qquad (2)$$

$$\frac{dz}{dt} - \lambda_3 z = 0 \qquad (3)$$

Step 1. The solution to Eq. (3) is

$$z(t) = c_3 e^{\lambda_3 t}$$

Step 2. Insert this expression into Eq. (2) and obtain

$$\frac{dy}{dt} - \lambda_2 y = \gamma c_3 e^{\lambda_3 t} \qquad (4)$$

The solution to this equation is

$$y(t) = c_2 e^{\lambda_2 t} + y_p(t)$$

where $y_p(t)$ is a particular solution to Eq. (4).

Step 3 Insert $y(t)$ into Eq. (1) on page 37.1 and obtain

$$\frac{dx}{dt} - \lambda_1 x = \alpha\left[c_2 e^{\lambda_2 t} + y_p(t)\right] + \beta c_3 e^{\lambda_3 t} \qquad (5)$$

Its solution is

$$x = c_1 e^{\lambda_1 t} + x_p(t)$$

where $x_p(t)$ is a particular solution to Eq. (5).

Conclusion:

The evolution of a dynamical system, Eq. (*) on page 37.1, generated by an upper triangular matrix has three degrees of freedom

$$x(t) = c_1 e^{\lambda_1 t} + x_p(t)$$
$$y(t) = c_2 e^{\lambda_2 t} + y_p(t)$$
$$z(t) = c_3 e^{\lambda_3 t}$$

The z-degree of freedom is completely independent of the others. The y-degree of freedom, although it characterized by $e^{\lambda_2 t}$, is influenced by $z(t)$ whenever $\gamma \neq 0$. The x-degree of freedom, $\propto e^{\lambda_1 t}$, is influenced by both y and z whenever $\alpha \neq 0, \beta \neq 0$.

It follows that the degrees of freedom of an upper triangle-generated time invariant linear systems have the hierarchical relationship

$$x(t) = c_1 e^{\lambda_1 t} + x_p(t)$$

↑

$$y(t) = c_2 e^{\lambda_2 t} + y_p(t)$$

↑

$$z(t) = c_3 e^{\lambda_3 t}$$

B.) General Time Invariant Linear System

It turns out that <u>any</u> linear system

$$\frac{d\vec{u}}{dt} = A\vec{u}$$

is similar to one generated by a triangular matrix, defective or non-defective, such as in Eq.(*) on page 37.1

C) The Triangularization Theorem

Moreover, this can always be done by means of a <u>unitary transformation</u>.

These claims are made precise by

the following

Theorem 37.1 (Read 5.6 in Strang 3rd Ed'n)

Let

$$A : C^n \longrightarrow C^n$$

be a square matrix.

Conclusion: ∃ a unitary $M = U_1 U_2 \cdots U_{n-1} = U$

such that

$$U_{n-1}^{-1} \cdots U_1^{-1} A U_1 \cdots U_{n-1} = (U_1 \cdots U_{n-1})^{-1} A U_1 \cdots U_{n-1} = U^{-1} A U = U^{H} A U$$

is upper triangular:

$$U^{-1} A U = T = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} ; \quad A = U T U^{-1}$$

D) Applications of the Triangularization Theorem

This theorem has nontrivial consequences.

Lemma 37.1

If A is $\left\{ \begin{array}{l} \text{Hermetian} \\ \text{Skew-Hermetian} \\ \text{Unitary} \end{array} \right.$

then

$U^{-1} A U$ is $\left\{ \begin{array}{l} \text{Hermetian} \\ \text{skew-Hermetian} \\ \text{Unitary} \end{array} \right.$

proof:

$$(U^{-1}AU)^H = U^H A^H (U^{-1})^H$$
$$= U^{-1} A^H U$$

Thus

$$(U^{-1}AU)^H = U^{-1}AU \quad \text{if } A^H = A;$$
$$(U^{-1}AU)^H = -U^{-1}AU \quad \text{if } A^H = -A;$$
$$(U^{-1}AU)^H = U^{-1}A^{-1}U \quad \text{if } A^H = A^{-1}$$

Mply by $U^{-1}AU$:

$$(U^{-1}AU)(U^{-1}AU)^H = (U^{-1}AU)(U^{-1}A^{-1}U)$$
$$= I$$

$(U^{-1}AU)^H = (U^{-1}AU)^{-1}$, which is the unitarity property.

Lemma 37.2

$$A \left\{ \begin{matrix} \text{Hermetian} \\ \text{skew- ''} \\ \text{unitary} \end{matrix} \right\} \text{matrix which is}$$

upper triangular is diagonal

proof:

$$T = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}; \quad T^H = T \Rightarrow T^H = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & 0 & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$T^H = -T \Rightarrow T^H = \begin{bmatrix} -\bar{\lambda}_1 & 0 & 0 \\ 0 & -\bar{\lambda}_2 & 0 \\ & & \\ 0 & 0 & -\bar{\lambda}_n \end{bmatrix}$$

$$T^H = T^{-1} \Rightarrow \overbrace{\begin{bmatrix} \bar{\lambda}_1 & 0 & \cdots & 0 \\ \bar{*} & \bar{\lambda}_2 & \cdots & 0 \\ \bar{*} & \bar{*} & \cdots & \bar{\lambda}_n \end{bmatrix}}^{T^H} \overbrace{\begin{bmatrix} \lambda_1 & * & * & \cdots & * \\ 0 & \lambda_2 & \star & \cdots & \star \\ 0 & 0 & & \cdots & \lambda_n \end{bmatrix}}^{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$|\lambda_1|^2 = 1; \quad \bar{\lambda}_1 * = 0 \Rightarrow * = 0 \ \forall \text{elements } *$

$|\lambda_2|^2 = 1 \quad \lambda_2 \star = 0 \Rightarrow \star = 0 \ \forall \text{elements } \star$

$\vdots$

$|\lambda_n|^2 = 1$

Thus $T = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}$

Corollary 37.1

Any $\left\{ \begin{array}{r} \text{Hermetian} \\ \text{skew-Hermetian} \\ \text{unitary} \end{array} \right\}$ matrix can be

diagonalized.

E.) Proof of the Triangularization Theorem

proof of Theorem 37.1 on page 37.6
(Triangular form via unitary U)

Step I

The matrix A has a least one eigenvector of length one:

$$x_1 : \quad A x_1 = \lambda_1 x_1$$

a) With x_1 as its 1st column construct, using the G-S process, if necessary, a matrix U_1 with orthonormal columns.

b) calculate $U_1^{-1} A U_1 \equiv A_1$

One therefore obtains

$$A\underbrace{\begin{bmatrix} | & & | \\ x_1 & * \cdots & * \\ | & & | \end{bmatrix}}_{\substack{U_1\\ \uparrow \\ \text{G-S}}} = \underbrace{\begin{bmatrix} | & \\ \lambda_1 x_1 & \cdots \\ | & \end{bmatrix}}_{AU_1} = \underbrace{\begin{bmatrix} | & | & | \\ x_1 & * \cdots & * \\ | & | & | \end{bmatrix}}_{U_1} \underbrace{\begin{bmatrix} \lambda_1 & * \cdots & * \\ 0 & * & * \\ \vdots & & \\ 0 & * & * \end{bmatrix}}_{\substack{U_1^{-1}AU \\ \uparrow \\ \text{calculated}}}$$

Notice that because the 1st column:

of U_1 is an eigen vector of A, it necessarily follows that in $U_1^{-1} A U_1$ all elements below λ_1 in the 1st column are zero.

Thus the 1st step in the triangularization process yields the partially triangularized matrix

$$U_1^{-1} A U_1 = \left[\begin{array}{c|ccc} \lambda_1 & * & & * \\ \cline{2-4} 0 & * & & * \\ \vdots & & & \\ 0 & * & & * \end{array}\right] \equiv A_1 \qquad (\star)$$

$$\underbrace{\qquad\qquad}_{M_2}$$

Step II. Apply the same kind of reasoning to the $(n-1)\times(n-1)$ submatrix M_2 that was used on A, namely, for M_2 find an eigen vector x_2 of length one:

$$x_2 : M_2 X_2 = \lambda_2 X_2$$

a) With x_2 as its 1st column construct, using the G-S process, if necessary, an $(n-1)\times(n-1)$ unitary matrix with orthonormal column. This matrix, when augmented with zeroes and 1 on the upper left, yields a $n\times n$ matrix U_2 which is also unitary

b) Calculate $U_2^{-1} A_1 U_2 \equiv A_2$

One therefore obtains

$$A_1 \underbrace{\begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \underbrace{\begin{matrix} | & & | \\ x_2 & \cdots & * \\ | & & | \end{matrix}}_{\substack{(n-1)(n-1) \\ \text{unitary}}} \\ 0 & \end{bmatrix}}_{\substack{U_2 \text{ is} \\ \text{unitary}}} = \underbrace{\begin{bmatrix} \lambda_1 & * & \\ 0 & | & \\ \vdots & \lambda_2 x_2 & \cdots \\ 0 & | & \end{bmatrix}}_{A_1 U_2} = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & | & & | \\ & x_2 & \cdots & * \\ 0 & | & & | \end{bmatrix}}_{\substack{U_2 \text{ is} \\ \text{unitary}}} \underbrace{\begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & & * \\ 0 & 0 & * & * \\ \vdots & \vdots & & \\ 0 & 0 & * & * \end{bmatrix}}_{U_2^{-1} A_1 U_2 \equiv A_2}$$

Again notice that, because the 2nd column of U_2 is an eigen vector of

$$A_1 = \left[\begin{array}{c|ccc} \lambda_1 & * & \cdots & * \\ \hline 0 & & & \\ 0 & & M_2 & \\ 0 & & & \end{array}\right] \equiv U_1^{-1} A U_1$$

in the previous step, Eq. (★) page 37.10, it necessarily follows that in

$$U_2^{-1} A_1 U \equiv A_2$$

all elements below λ_1 and λ_2 in the 1st and 2nd column are zero.

Thus the 2nd step in the triangularization process yields the partially triangularized matrix

$$\begin{bmatrix} \lambda_1 & * & * \cdots & * \\ 0 & \lambda_2 & * & * \\ 0 & 0 & * & * \\ \vdots & \vdots & \vdots & \\ 0 & 0 & * & * \end{bmatrix} = U_2^{-1} A_1 U_2 = U_2^{-1} U_1^{-1} A U_1 U_2 \equiv A_2$$

continuing this process one arrives at

Step $n-1$

$$A_{n-1} = \overset{U^{-1}}{\overbrace{U_{n-1}^{-1} \cdots U_1^{-1}}} A \overset{U}{\overbrace{U_1 \cdots U_{n-1}}}$$

$$= \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & & \\ 0 & 0 & \ddots & \\ \vdots & & \ddots & * \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = T$$

Conclusion

$$T = U^{-1} A U$$

i.e.

any $n \times n$ matrix A is unitarily similar to an upper triangular matrix T.