LECTURE 38

NORMAL MATRICES

Normality $\iff$ Diagonalizability via unitary similarity transformation.
NORMAL MATRICES

The difference between Hermitian, unitary, and skew-Hermitian matrices lies in where their eigenvalues are located in the complex $\lambda$-plane.

$K^H = -K$
$U^H = U^{-1}$

Figure 38.1: Locus of eigenvalues $\lambda$ of Hermitian, unitary, and skew-Hermitian matrices.

All such matrices are diagonalizable by means of appropriate unitary similarity transformations.

Question: Are there any other matrices which are diagonalizable by unitary transformations? If so, what are they?

Answer: Yes. They are precisely the normal matrices, and no others, which are diagonalizable by a unitary transformation. Their eigenvalues can lie anywhere in the complex $\lambda$-plane.

This answer is put into mathematical terms by means of the following definition and theorem:

**Definition 38.1 (Normal Matrix)**

A matrix $N$ is said to be normal if it commutes with its Hermitian adjoint:

$$NN^H - N^HN = [N, N^H] = 0$$

**Note here:** $[N, N^H]$ is called the commutator of $N$ and $N^H$.

(i) if $[N, N^H] = 0$, one says that $N$ commutes with $N^H$.
Theorem 38.1 (Normality \implies Diagonalizability
via unitary transform)

(a) For normal matrices, and no others,

\exists \text{ unitary } U \text{ such that }

T = U^{-1}NU = N (= \text{diagonal matrix})

i.e.

(b) Normal matrices are exactly those with a complete set of orthogonal eigenvectors.

\quad i.e.

(c) \quad N \text{ is normal } \iff \text{N is diagonalizable by a unitary } U.

Proof: \iff

\text{N is diagonalizable means }

N = U\Lambda U^{-1}

N^H = (U\Lambda U^{-1})^H = (U^{-1})^H \Lambda^H U^{-1}

= U^H \Lambda U^{-1}

\text{NN}^H = U\Lambda U^{-1} U^H \Lambda U^{-1} = U\Lambda \Lambda^H U^{-1}

\text{N}^HN = U^H \Lambda U^{-1} U\Lambda U^{-1} = U \Lambda \Lambda U^{-1} U^{-1}

\text{NN}^H - N^HN = U(\Lambda^H - \Lambda \Lambda)U^{-1} = 0, \quad \text{Q.E.D.}

In order to show normality \implies diagonalizability via unitary U, we first show that upper triangularization (see Theorem 37.1 on p. 376) preserved normality, and then proceed to show that an upper triangular normal matrix is diagonal.

Comment:

Normality is preserved by a unitary similarity transformation. Indeed let

T = U^{-1}NU; \quad T^H = (U^{-1}NU)^H = U^H N^H (U^{-1})^H

so that

TT^H - T^HT = U^{-1}NU U^{-1}N^H U

= U^{-1} (NN^H - N^H N) U

\begin{align*}
&= U^{-1} (NN^H - N^H N) U \\
&= 0
\end{align*}

Thus, T is also normal indeed.
proof: \[ \Rightarrow \]

Having brought $A$ into triangular form by means of a unitary similarity transformation $T = U^*AU$.

We now show that $T$ is diagonal.

$T^H T = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ * & \lambda_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \lambda_m \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{bmatrix} = \begin{bmatrix} |\lambda_1|^2 & * & \cdots & * \\ * & |\lambda_2|^2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & |\lambda_m|^2 \end{bmatrix}$

Thus $T^H T = \begin{bmatrix} |\lambda_1|^2 + |\lambda_2|^2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & |\lambda_m|^2 + |\lambda_m|^2 \end{bmatrix}$

Next $\left( T^H T \right)_{1,1} = (T^H)_{1,1} \Rightarrow (T^H T)_{2,2} = (T^H T)_{1,1} \Rightarrow |\lambda_2|^2 + |\lambda_2|^2 = |\lambda_2|^2 \Rightarrow |\lambda_2|^2 + |\lambda_2|^2 = |\lambda_2|^2$

Thus beyond $\lambda_1$, the second row of $T$ is zero $\Rightarrow d_2 = 0$

$T^H T = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ * & \lambda_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \lambda_m \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{bmatrix} = \begin{bmatrix} |\lambda_1|^2 & * & \cdots & * \\ * & |\lambda_2|^2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & |\lambda_m|^2 \end{bmatrix}$

Element-wise equality of $T^H T$ implies

$(T^H T)_{1,1} = (T^H T)_{1,1} \Rightarrow |\lambda_1|^2 + |\lambda_2|^2 = |\lambda_1|^2$

Consequently $c_1 = [\cdots] = 0 \Rightarrow \cdots$

Element-wise equality of $T^H T$ implies

$T = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{bmatrix}$

Thus beyond $\lambda_1$, the $(n-1)^{th}$ row of $T$ is zero $\Rightarrow d_3 = 0$

Thus all the off-diagonal elements of $T$

$c_1, \ldots, c_m$ vanish. Consequently

$T = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{bmatrix}$

Q.E.D.