LECTURE 38

NORMAL MATRICES

Normality $\Longleftrightarrow$ Diagonalizibility via unitary similarity transformation.

NORMAL MATRICES

The difference between Hermetian, unitary, and skew-Hermetian matrices lies in where their eigenvalues are located in the complex λ-plane

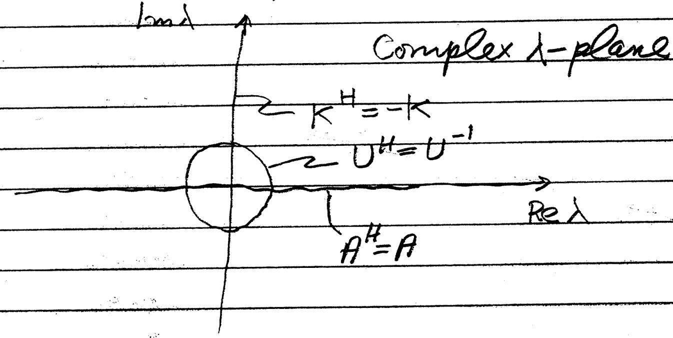

Figure 38.1
Locus of eigenvalues λ of Hermetian, unitary, and skew-Hermetian matrices.

All such matrices are diagonalizable by means of appropriate unitary similarity transformations.

Question: Are there any other matrices which are diagonalizable by unitary transformations? If so what are they?

Answer: Yes. They are precisely the normal matrices, and no others, which are diagonalizable by a unitary transformation

Their eigenvalue can lie anywhere in the complex λ-plane.

This answer is put into mathematical terms by means of the following definition and theorem:

Definition 38.1 (Normal Matrix)

A matrix N is said to be **normal** if it commutes with its Hermetian adjoint:

$$NN^H - N^H N \equiv [N, N^H] = 0$$

Nota bene: (i) $[N, N^H]$ is called the commutator of N and N^H.
(ii) $[N, N^H] = 0$ one says that N commutes with N^H

<u>Theorem</u> 38.1 (Normality $\Rightarrow$ Diagonalizibility via unitary transform)

(a) For normal matrices, and no others, $\exists$ unitary U such that

$$T = U^{-1} N U = \Lambda (\equiv \text{diagonal matrix})$$

i.e

(b) Normal matrices are exactly those with a complete set of orthonormal eigenvectors

i.e

(c) N is normal $\Leftrightarrow$ N is diagonalizable by a unitary U.

<u>Proof:</u> $\Leftarrow$

N is diagonalizable means

$$N = U \Lambda U^{-1}$$

$$N^H = (U\Lambda U^{-1})^H = (U^{-1})^H \Lambda^H U^H = U \Lambda^H U^{-1}$$

$$NN^H = U\Lambda U^{-1} U \Lambda^H U^{-1} = U \Lambda \Lambda^H U^{-1}$$

$$N^H N = U \Lambda^H U^{-1} U \Lambda U^{-1} = U \Lambda^H \Lambda U^{-1}$$

$$NN^H - N^H N = U(\Lambda\Lambda^H - \Lambda^H \Lambda) U^{-1} = 0, \qquad \text{Q.E.D.}$$

In order to show normality $\Rightarrow$ diagonalizibility via unitary U, we first show that upper triangularization (see Theorem 37.1 on p 37.6) preserves normality, and then proceed to show that an upper triangular normal matrix is diagonal.

<u>Comment:</u>

Normality is preserved by a unitary similarity transformation. Indeed

Let

$$T = U^{-1} N U \; ; \; T^H = (U^{-1} N U)^H = U^H N^H (U^{-1})^H$$

so that

$$TT^H - T^H T = U^{-1} N U \, U^{-1} N^H U - U^{-1} N^H U \, U^{-1} N U = U^{-1} (\underbrace{NN^H - N^H N}_{\text{zero}}) U = 0$$

Thus T is also normal indeed.

proof: $\Longrightarrow$

Having brought N into triangular form by means of a unitary similarity transformation

$$T = U^{-1} N U,$$

we now show that T is diagonal.

$$T T^H = \begin{bmatrix} \lambda_1 & \overbrace{* \cdots *}^{\vec{c}_1} \\ 0 & \lambda_2 & & * \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \bar\lambda_1 & 0 & \cdots & 0 \\ * & \bar\lambda_2 & & 0 \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & \bar\lambda_n \end{bmatrix} = \begin{bmatrix} |\lambda_1|^2 + |\vec{c}_1|^2 & & \\ & |\lambda_2|^2 + |\vec{c}_2|^2 & \\ & & \ddots \\ & & & |\lambda_n|^2 + 0 \end{bmatrix}$$

$$T^H T = \begin{bmatrix} \bar\lambda_1 & 0 & \cdots & 0 \\ * & \bar\lambda_2 & & 0 \\ \vdots & \vdots & & \\ * & * & & \bar\lambda_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \overbrace{* \cdots *}^{\vec{c}_1} \\ 0 & \lambda_2 & & * \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} |\lambda_1|^2 & & \\ & |\lambda_2|^2 + |d_2|^2 & \\ & & \ddots \\ & & & |\lambda_n|^2 + \cdots \end{bmatrix}$$

Element-wise equality of $T^H T$ implies

$$(T T^H)_{11} = (T^H T)_{11}$$

$$|\lambda_1|^2 + |\vec{c}_1|^2 = |\lambda_1|^2$$

Consequently

$$\vec{c}_1 = [* \cdots *] = \overrightarrow{\text{zero}}$$

i.e. $|\vec{c}_1|^2 = 0 \Rightarrow$ beyond λ_1 the 1st row of T is zero $\Rightarrow$ $d_2 = 0$

Thus $$T^H T = \begin{bmatrix} |\lambda_1|^2 & & \\ & +|\lambda_2|^2 & \\ & & |\lambda_3|^2 + |d_3|^2 \\ & & & \ddots \end{bmatrix}$$

Next

$$(T T^H)_{22} = (T^H T)_{22}$$

$$|\lambda_2|^2 + |\vec{c}_2|^2 = |\lambda_2|^2$$

$\Rightarrow |\vec{c}_2|^2 = 0$

Thus beyond λ_2 the 2^{nd} row of T is zero $\Rightarrow$ $d_3 = 0$

$\vdots$

$\Rightarrow |\vec{c}_{n-1}|^2 = 0$

Thus beyond λ_{n-1} the $(n-1)^{st}$ row of T is zero $\Rightarrow$

Thus all the off-diagonal elements of T, $\vec{c}_1, \ldots, \vec{c}_{n-1}$ vanish. Consequently,

$$T = \begin{bmatrix} \lambda_1 & & O \\ & \lambda_2 & \\ O & & \ddots \\ & & & \lambda_n \end{bmatrix}$$

QED.