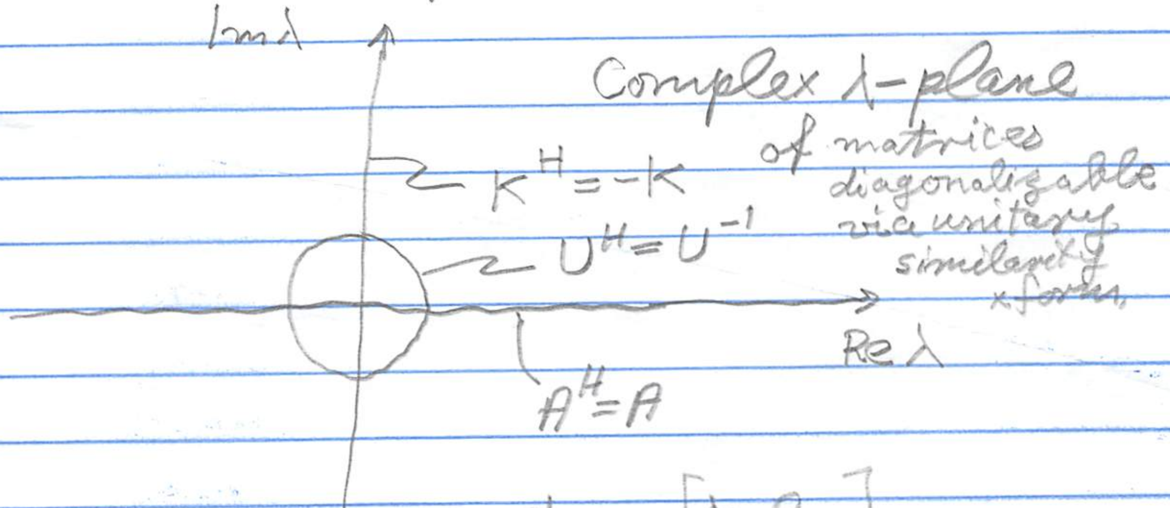
LECTURE 38

NORMAL MATRICES

Normality $\Longleftrightarrow$ Diagonalizibility via unitary similarity transformation.

NORMAL MATRICES

The difference between Hermetian, unitary, and skew-Hermetian matrices lies in where their eigenvalues are located in the complex λ-plane



$$U^{-1}AU = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Figure 38.1
Locus of eigenvalues λ of Hermetian, unitary, and skew-Hermetian matrices.

All such matrices are diagonalizable by means of appropriate unitary similarity transformations.

Question: Are there any other matrices which are diagonalizable by unitary transformations? If so what are they?

Answer: Yes. They are precisely the normal matrices, and no others, which are diagonalizable by a unitary transformation

Their eigenvalue can lie anywhere in the complex λ-plane.

This answer is put into mathematical terms by means of the following definition and theorem:

Definition 38.1 (Normal Matrix)

A matrix N is said to be normal if it commutes with its Hermetian adjoint:

$$NN^H - N^H N \equiv [N, N^H] = 0$$

Nota bene: (i) $[N, N^H]$ is called the commutator of N and N^H.
(ii) $[N, N^H] = 0$ one says that N commutes with N^H

$\Leftrightarrow$ not merely $\Rightarrow$ **Theorem 38.1** (Normality $\Leftrightarrow$ Diagonalizibility via unitary transform)

(a) For normal matrices, and no others,

$\exists$ unitary U such that

$$T = U^{-1} N U \equiv \Lambda \quad (\equiv \text{diagonal matrix})$$

i.e

(b) Normal matrices are exactly those with a complete set of orthonormal eigenvectors

i.e

(c) N is normal $\Leftrightarrow$ N is diagonalizable by a unitary U.

Proof: $\Leftarrow$

N is diagonalizable means

$$N = U \Lambda U^{-1}$$

$$N^H = (U \Lambda U^{-1})^H = (U^{-1})^H \Lambda^H U^H -$$

$$= U \Lambda^H U^{-1}$$

$$N N^H = U \Lambda U^{-1} U \Lambda^H U^{-1} = U \Lambda \Lambda^H U^{-1}$$

$$N^H N = U \Lambda^H U^{-1} U \Lambda U^{-1} = U \Lambda^H \Lambda U^{-1}$$

$$N N^H - N^H N = U (\Lambda \Lambda^H - \Lambda^H \Lambda) U^{-1} = 0, \qquad \text{Q.E.D.}$$

In order to show normality $\Rightarrow$ diagonalizibility via unitary U, we first show that upper triangularization (see Theorem 37.1 on p 37.6) preserves normality, and then proceed to show that an upper triangular normal matrix is diagonal.

Comment:

Normality is preserved by a unitary similarity transformation. Indeed

Let

$T = U^{-1}NU$; $T^H = (U^{-1}NU)^H = U^H N^H (U^{-1})^H$

so that

$$\begin{aligned} TT^H - T^H T &= U^{-1}NU\ U^{-1}N^H U \\ &\quad - U^{-1}N^H U\ U^{-1}NU \\ &= U^{-1}\underbrace{(NN^H - N^H N)}_{\text{zero}}U \\ &= 0 \end{aligned}$$

Thus T is also normal indeed.

proof: $\Longrightarrow$

Having brought N into triangular form, by means of a unitary similarity transformation

$$T = U^{\dagger} N U,$$

we now show that T is diagonal.

$$TT^H = \begin{bmatrix} \lambda_1 & \overbrace{* \cdots *}^{\vec{c}_1} \\ 0 & \lambda_2 & & * \\ \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \bar{\lambda}_1 & 0 & \cdots & 0 \\ \bar{*} & \bar{\lambda}_2 & & 0 \\ \vdots & \vdots & & \vdots \\ \bar{*} & \bar{*} & \cdots & \bar{\lambda}_n \end{bmatrix} = \begin{bmatrix} |\lambda_1|^2 + |c_1|^2 \\ & |\lambda_2|^2 + |c_2| \\ & & \ddots \\ & & & |\lambda_n|^2 + \underline{0} \end{bmatrix}$$

$$T^H T = \begin{bmatrix} \bar{\lambda}_1 & 0 & \cdots & 0 \\ \bar{*} & \bar{\lambda}_2 & & 0 \\ \vdots & \vdots & & \vdots \\ \bar{*} & \bar{*} & & \bar{\lambda}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \overbrace{* \cdots *}^{\vec{c}_1} \\ 0 & \lambda_2 & & * \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} |\lambda_1|^2 \\ & |\lambda_2|^2 + |d_2|^2 \\ & & \ddots \\ & & & |\lambda_n|^2 + \cdots \end{bmatrix}$$

zero

zero

Element-wise equality of $T^H T$ implies

$$(TT^H)_{11} = (T^H T)_{11}$$

$$|\lambda_1|^2 + |\vec{c}_1|^2 = |\lambda_1|^2$$

Consequently

$$\vec{c}_1 = [* \cdots *] = \overrightarrow{\text{zero}}$$

$$TT^H = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & * \\ \vdots & & & \\ 0 & 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \bar{\lambda}_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & * & \cdots & \bar{\lambda}_n \end{bmatrix} = \begin{matrix} |\lambda_1|^2 & 0 & \cdots & 0 \\ 0 & |\lambda_2|^2 + |d_2|^2 & & \end{matrix}$$

$$T^H T = \begin{bmatrix} \bar{\lambda}_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \vdots & \vdots & \\ 0 & * & \bar{\lambda}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & * \\ & \vdots & \vdots \\ 0 & 0 & \lambda_n \end{bmatrix} =$$

i.e.,
$|c_1|^2 = 0 \Rightarrow$ beyond λ_1 the 1st row of T is zero $\Rightarrow$
$d_2 = 0$

Thus $$T^H T = \begin{bmatrix} |\lambda_1|^2 & 0 & 0 & \cdots & 0 \\ & +|\lambda_2|^2 & \overbrace{\cdots}^{c_2} & * & \\ & & |\lambda_3|^2+|d_3|^2 & & \\ & & & \ddots & \end{bmatrix}$$

Next
$$(TT^H)_{22} = (T^H T)_{22}$$
$$|\lambda_2|^2 + |\vec{c}_2|^2 = |\lambda_2|^2$$
$\Rightarrow |\vec{c}_2|^2 = 0$

Thus beyond λ_2 the 2nd row of T is zero $\Rightarrow$
$d_3 = 0$

$\vdots$

$\Rightarrow |\vec{c}_{n-1}|^2 = 0$

Thus beyond λ_{n-1} the $(n-1)$st row of T is zero $\Rightarrow$

Thus all the off-diagonal elements of T, $\vec{c}_1, \ldots, \vec{c}_{n-1}$ vanish. Consequently,

$$T = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix}$$

Q.E.D.

i.e. a triangularized normal matrix is a diagonal matrix.