LECTURE 38

NORMAL MATRICES

Normality $\iff$ Diagonalizibility via unitary similarity transformation.
NORMAL MATRICES

The difference between Hermitian, unitary, and skew-Hermitian matrices lies in where their eigenvalues are located in the complex $\lambda$-plane.

Complex $\lambda$-plane

$K^H = -K$

$U^H = U^{-1}$

$A^H = A$

$U^H A U = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_m \end{bmatrix}$

Figure 38.1

Locus of eigenvalues $\lambda$ of Hermitian, unitary, and skew-Hermitian matrices.

All such matrices are diagonalizable by means of appropriate unitary similarity transformations.
Question: Are there any other matrices which are diagonalizable by unitary transformations? If so what are they?

Answer: Yes. They are precisely the normal matrices and no others, which are diagonalizable by a unitary transformation. Their eigenvalues can lie anywhere in the complex $\lambda$-plane.

This answer is put into mathematical terms by means of the following definition and theorem:

**Definition 38.1 (Normal Matrix)**

A matrix $N$ is said to be normal if it commutes with its Hermitian adjoint:

$$NN^* - N^*N = [N, N^*] = 0$$

_Nota bene:_ (i) $[N, N^*]$ is called the commutator of $N$ and $N^*$.

(ii) $[N, N^*] = 0$ one says that $N$ commutes with $N^*$.
Theorem 38.1 (Normality $\iff$ Diagonalizability via unitary transform)

(a) For normal matrices, and no others, there exists a unitary $U$ such that

$$T = U^{-1}NU = \Lambda \quad (\equiv \text{diagonal matrix})$$

i.e.

(b) Normal matrices are exactly those with a complete set of orthonormal eigenvectors

i.e.

(c) $N$ is normal $\iff$ $N$ is diagonalizable by a unitary $U$.

Proof:

$N$ is diagonalizable means

$$N = U\Lambda U^{-1}$$

$$N^H = (U\Lambda U^{-1})^H = (U^{-1})^H \Lambda^H U^H = U \Lambda^H U^{-1}$$

$$NN^H = U \Lambda U^{-1} U \Lambda^H U^{-1} = U \Lambda \Lambda^H U^{-1}$$

$$N^HN = U \Lambda^H U^{-1} U \Lambda U^{-1} = U \Lambda \Lambda^H U^{-1}$$

$$NN^H - N^HN = U \left(\Lambda \Lambda^H - \Lambda^H \Lambda\right) U^{-1} = 0, \quad \text{Q.E.D.}$$
In order to show normality $\Rightarrow$ diagonalizability via unitary $U$, we first show that upper triangularization (see Theorem 37.1 on p. 37.6) preserved normality, and then proceed to show that an upper triangular normal matrix is diagonal.

Comment:
Normality is preserved by a unitary similarity transformation. Indeed let
\[ T = U^{-1}NU; \quad T^H = (U^{-1}NU)^H = U^HN^H(U^{-1})^H \]
so that
\[ T^H - T^HT = U^{-1}NU - U^{-1}N^HU \]
\[ = U^{-1}(NN^H - N^HN)U \]
\[ = 0 \]
Thus $T$ is also normal indeed.
Having brought $N$ into triangular form by means of a unitary similarity transformation:

$$T = U^*NU,$$

we now show that $T$ is diagonal.

\[
\begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{bmatrix}
= \begin{bmatrix}
|\lambda_1|^2 + |\xi|^2 \\
|\lambda_2|^2 + |\xi|^2 \\
\vdots \\
|\lambda_n|^2 + |\xi|^2
\end{bmatrix}.
\]

$$T^HT = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{bmatrix}
= \begin{bmatrix}
|\lambda_1|^2 \\
|\lambda_2|^2 \\
\vdots \\
|\lambda_n|^2
\end{bmatrix} + |\xi|^2 \mathbf{1}^T \mathbf{1}.
$$

Element-wise equality of $T^HT$ implies

\[
(T^HT)_{11} = (T^HT)_{11},
\]

$$|\lambda_1|^2 + |\zeta|^2 = |\lambda_1|^2,$$

consequently \[
\hat{\zeta} = [* \cdots *] = \text{zero}.
\]
\[ TT^H = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m^2 \end{bmatrix} \]

\[ T^H T = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m^2 \end{bmatrix} \]
Thus $T^{H}T = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ \frac{\lambda_2^2}{\lambda_3^2 + d_3^2} & \frac{\lambda_2}{\lambda_3 + d_3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Next $(TT^{H})_{22} = (T^{H}T)_{22}$

$
\frac{\lambda_2^2}{\lambda_3^2 + d_3^2} + \frac{\Sigma_{2} \Sigma_{1}}{\lambda_3^2 + d_3^2} = \lambda_3^2
$

$\Rightarrow |c_2|^2 = 0$

Thus beyond $\lambda_2$ the second row of $T$ is zero $\Rightarrow d_3 = 0$

$\Rightarrow |c_{m-1}|^2 = 0$

Thus beyond $\lambda_{m-1}$ the $(n-1)^{st}$ row of $T$ is zero $\Rightarrow$

Thus all the off-diagonal elements of $T$

$c_1, c_2, \ldots, c_{m-1}$ vanish. Consequently

$T = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

QED.

i.e., a triangularized normal matrix is a diagonal matrix.