LECTURE 39

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I) THE UNIVERSE, MATHEMATICS AND QUADRATIC FORMS

The complexity of mathematics is a reflection of the complexity of the relationships that exist in the universe.

Progress in science and engineering necessarily depends on identifying these relationships and then casting them into numerical, i.e. mathematical, form.

A significant part of these relationship is conceptualized in terms of multi-variable scalar functions $F(x^1, x^2, \ldots, x^n) \equiv F(\vec{x})$ represented in terms of a Taylor

$$F(\vec{x}) = F(\vec{0}) + \frac{\partial F}{\partial x^i} x^i + \frac{1}{2} x^i \frac{\partial^2 F}{\partial x^i \partial x^j} x^j + \cdots$$

$$\boxed{F(\vec{x}) = F(\vec{0}) + \langle \vec{b}, \vec{x} \rangle + \langle \vec{x}, A\vec{x} \rangle + \cdots}$$

and the linear mathematics surrounding it.

Of particular physical interest is the circumstance where $\vec{x} = \vec{0}$ is a critical point, which is characterized by

$$\frac{\partial F(\vec{0})}{\partial x^i} = 0.$$

In that case the local behaviour of $F(\vec{x})$ in the neighborhood of this point is controlled entirely by the symmetric matrix

$$[A_{ij}] = \left[\frac{1}{2} \frac{\partial^2 F(\vec{0})}{\partial x^i \partial x^j} \right]$$

Thus the study of $F(\vec{x}) = F(\vec{0}) + \langle \vec{x}, A\vec{x} \rangle + \cdots$ reduces to a study of quadratic forms $\langle \vec{x}, A\vec{x} \rangle$.

Side Comment (Q: Need A be symmetric? A: YES!)

In studying a quadratic form

$$\langle x, Ax\rangle = x^i A_{ij} x^j$$

one might ask: Can A be antisymmetric?

The answer is NO. Indeed, consider

$$x^i A_{ij} x^j = \frac{1}{2} x^i (A_{ij} + A_{ji}) x^j + \frac{1}{2} x^i (A_{ij} - A_{ji}) x^j.$$

In the second double sum let $i = j'$ and $j = i'$, and then drop the "prime":

$$x^i A_{ij} x^j = \frac{1}{2} x^i (A_{ij} + A_{ji}) x^j + \frac{1}{2} x^i A_{ij} x^j - \frac{1}{2} x^{j'} A_{i'j'} x^{i'}$$

$$= \quad '' \quad\quad '' \quad - \frac{1}{2} x^{i'} A_{i'j'} x^{j'}$$

$$= \quad '' \quad + \underbrace{\frac{1}{2} x^i A_{ij} x^j - \frac{1}{2} x^i A_{ij} x^j}_{\text{zero}}$$

Thus

$$\boxed{x^i A_{ij} x^j = \frac{1}{2} x^i (A_{ij} + A_{ji}) x^j}$$

One therefore arrives at the following

Conclusion:

(i) Only the symmetric part of A, namely $\frac{1}{2}(A + A^T)$, determines and is determined by the quadratic form.

(ii) The quadratic form $\langle x, Ax\rangle$ implies that $\exists$ a unique symmetric matrix A.

(iii) Given $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$, $\langle x, Ax\rangle$ is determined by $\frac{1}{2}(A + A^T)$; $\frac{1}{2}(A - A^T)$ contributes nothing.

II.) POSITIVE DEFINITE MATRICES

The function

$$F(\vec{x}) = F(\vec{0}) + \langle \vec{x}, A\vec{x} \rangle$$

has a minimum whenever

$$\boxed{\langle \vec{x}, A\vec{x} \rangle > 0 \qquad \forall \vec{x} \neq \vec{0}} \qquad (*)$$

Definition

Matrix A is said to be positive definite whenever condition $(*)$ is satisfied.

Example: $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

Consider

$$\langle \vec{x}, A\vec{x} \rangle = [x\, y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= a x^2 + 2bxy + cy^2 \equiv f$$

By completing the square one obtains

$$\langle \vec{x}, A\vec{x} \rangle = a\left(x + \frac{b}{a} y\right)^2 + \left(c - \frac{b^2}{a}\right) y^2$$

Conclusion:

$$\left.\begin{array}{l} i)\ a > 0 \\ ii)\ ac - b^2 > 0 \end{array}\right\} \iff A \text{ is positive definite}$$

or equivalently

$$\left.\begin{array}{l} i)\ a > 0 \\ ii)\ \det \begin{vmatrix} a & b \\ b & c \end{vmatrix} > 0 \end{array}\right\} \iff A \text{ is positive definite.}$$

Comment

a) A is said to be positive semi-definite if i) $a > 0$

ii) $ac - b^2 = 0$

In that case $x = -\frac{b}{a} y \Rightarrow \langle x, Ax \rangle = 0$

b) A is said to be indefinite if $ac - b^2 < 0$

In that case $\langle \vec{x}, A\vec{x} \rangle \gtreqless 0$ depending on x & y

For example if

(i) $a = c = 0$, then $\langle x, Ax \rangle = 2bxy$

(ii) $a = 1, c = -1, b = 0$, then $\langle x, Ax \rangle = x^2 - y^2$

III. THREE FACES OF POSITIVE DEFINITENESS

The positive definiteness of an $n \times n$ matrix A can be stated in three equivalent ways according to the following

Theorem

1. $x^T A x > 0 \quad \forall x$

2. All eigenvalues of A satisfy $\lambda_i > 0$

3. $\exists$ a non non-singular matrix W such that
$$A = W^T W$$

Proof! $1. \Rightarrow 2.$

Let x_i : $A x_i = \lambda_i x_i \quad i = 1, 2, \dots, n$

Then $0 < x_i^T A x_i = \lambda_i x_i^T x_i = \lambda_i \|x_i\|^2$

$2. \Rightarrow 1.$

Let $\{x_i\}$ be an O.N. eigenvector basis

$$x^T A x = (c_1 x_1^T + \dots + c_n x_n^T) A (c_1 x_1 + \dots + c_n x_n)$$
$$= (c_1 x_1^T + \dots + c_n x_n^T)(c_1 \lambda_1 x_1 + \dots + c_n \lambda_n x_n)$$
$$= c_1^2 \lambda_1^2 + \dots + c_n^2 \lambda_n^2 > 0$$

$2. \Rightarrow 3$

$$A = U \Lambda U^T = U \Lambda^{1/2} \Lambda^{1/2} U^T$$
$$= \underbrace{(\Lambda^{1/2} U^T)^T}_{W^T} \underbrace{\Lambda^{1/2} U^T}_{W}$$

Thus
$$A = W^T W$$

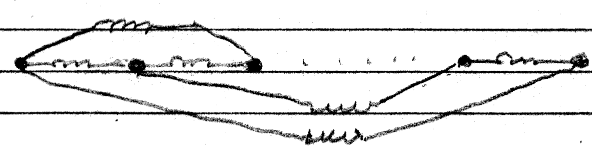
$3. \Rightarrow 1.$

$$A = W^T W : \quad x^T A x = x^T W^T W x$$
$$= (Wx)^T W x = \|Wx\|^2 > 0 \quad \forall x$$

IV) GEOMETRY OF QUADRATIC FORMS:

A. Vibrating System: Energy Conservation

Consider a linear system of n masses interconnected by springs



a) The state of these masses is expressed by their displacement away from equilibrium and their velocities

$$\begin{bmatrix} q^1(t) \\ \vdots \\ q^n(t) \end{bmatrix} \equiv q(t)\,;\quad \frac{d}{dt}\begin{bmatrix} q^1(t) \\ \vdots \\ q^n(t) \end{bmatrix} \equiv \dot{q}(t)$$

b) The multicomponent force of restitution is expressed in terms of the system's Hook's matrix A, which is symmetric:

$$\text{force}(t) = -\,Aq(t)\,.$$

c) Newton's eq'ns of motion for n identical unit masses is

$$\ddot{q}(t) + Aq(t) = 0 \qquad (*)$$

d) The system is time-invariant. Consequently, its total energy (T.E.) is constant. Indeed, multiplying on the left by the system velocity $\dot{q}^T$ one finds

$$\dot{q}^T\ddot{q} + \dot{q}^T A q = 0$$

or

$$\frac{d}{dt}\Big\{\underbrace{\underbrace{\tfrac{1}{2}(\dot{q}^T\dot{q})}_{\text{K.E.}} + \underbrace{\tfrac{1}{2}\,q^T A q}_{\text{P.E. = Potential Energy}}}_{\text{T.E}}\Big\} = 0$$

Thus

$$\text{T.E.} = \tfrac{1}{2}\|\dot{q}\|^2 + \tfrac{1}{2}\,q^T A q = \text{constant}$$

is a constant of motion, independent of time.

B. Vibrating System: Normal Modes & Their Energy

a) Consider a <u>normal mode</u>, a system motion where all masses have the same time-dependence $f(t)$ in their displacements,

$$q(t) = \begin{bmatrix} q^1(t) \\ \vdots \\ q^n(t) \end{bmatrix} = f(t) \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix} = f(t)\, x$$

For such a motion the equations of motion are

$$\ddot{f}(t)\, x = -f(t)\, A x$$

$$\frac{\ddot{f}}{f} x = -Ax \quad = \text{constant vector}.$$

Consequently, the time-dependence is

$$\frac{\ddot{f}}{f} = -\lambda \; ; \quad \boxed{\ddot{f} + \lambda f = 0} \qquad (1)$$

where λ is a constant whose value is such as to allow a non-trivial solution to

$$\boxed{Ax = \lambda x}$$

b) The general solution to Eq. (1) is

$$f(t) = c \sin(\omega t + \delta), \quad \omega = \sqrt{\lambda}$$

consequently the total energy carried by a normal mode is

$$T.E. = \frac{1}{2} \dot{q}^T \dot{q} + \frac{1}{2} q^T A q$$

$$= \frac{1}{2} \dot{f}^2 x^T x + \frac{1}{2} f^2 x^T A x$$

$$= \frac{1}{2} c^2 \lambda \cos^2(\omega t + \delta)\, x^T x + \frac{1}{2} c^2 \sin^2(\omega t + \delta) \underbrace{x^T A x}_{\lambda x^T x}$$

$$\boxed{T.E. = \frac{1}{2} c^2 \lambda = \frac{1}{2} c^2 \omega^2} \quad \left(\text{Energy in a normal mode}\right)$$

Thus the conserved T.E. of a normal mode is proportional to the eigenvalue

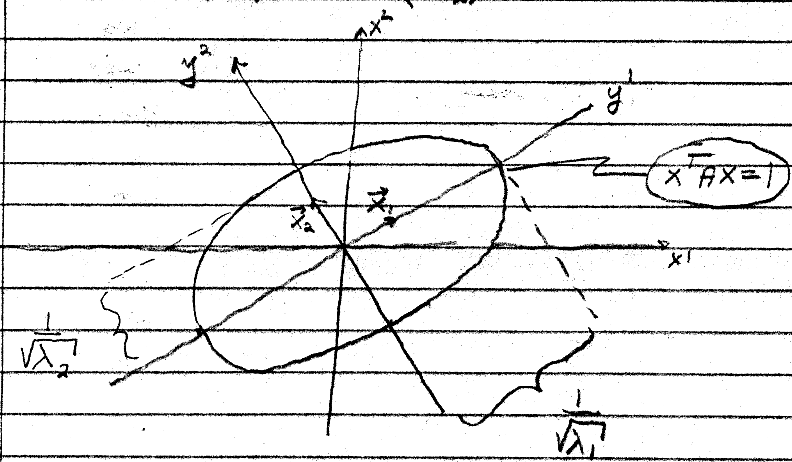
$$\lambda = \omega^2,$$

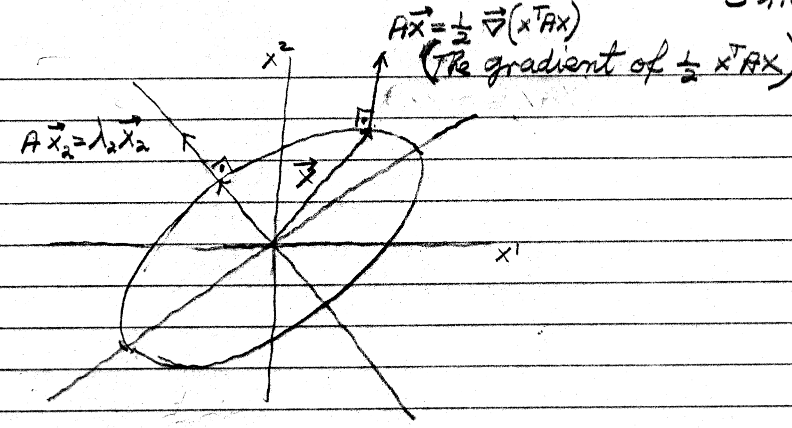
the squared (angular) frequency of the normal mode.

$$\boxed{\text{Thus the isograms } F = \text{const are concentric ellipsoids whose axes point along the direction of the eigenvectors } \vec{x}_1, \ldots, \vec{x}_n.}$$

In particular the $F=1$ ellipsoid in 2-D is

$$x^T A x = \frac{(y^1)^2}{\left(\frac{1}{\sqrt{\lambda_1}}\right)^2} + \frac{(y^2)^2}{\left(\frac{1}{\sqrt{\lambda_2}}\right)^2} = 1$$





The 1-1 correspondence between a symmetric matrix A and its quadratic form $x^T A x$ leads to the following conclusion:

A symmetric matrix with positive eigenvalues should be pictured as an ellipsoid (1) whose semimajor axes point along the eigenvectors of the eigenvector matrix

$$U = \begin{bmatrix} \vdots & & \vdots \\ x_1 & \cdots & x_n \\ \vdots & & \vdots \end{bmatrix}$$

and

② whose

$$(i^{th}\text{ semi-major axis})^2 = \frac{1}{\lambda_i}$$

E. Geometrization via Concentric Hyperboloids In 3 dimensions

If a) $0<\lambda_1<\lambda_2<\lambda_3$ $\quad F=1$ an ellipsoid

If b) $\lambda_1<0<\lambda_2<\lambda_3$

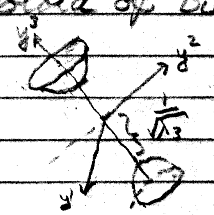
$$F = \frac{(y^3)^2}{\frac{1}{\lambda_3}} + \frac{(y^2)^2}{\frac{1}{\lambda_2}} - \frac{(y^1)^2}{\frac{1}{|\lambda_1|}} = 1$$

a hyperboloid of one sheet

If c) $\lambda_1<\lambda_2<0<\lambda_3$

$$F = \frac{(y^3)^2}{\frac{1}{\lambda_3}} - \frac{(y^2)^2}{\frac{1}{|\lambda_2|}} - \frac{(y^1)^2}{\frac{1}{|\lambda_1|}} = 1$$

a hyperboloid of two sheet



F. The Extremum Principle Geometrized.

The constraint, Eq.(2) on page 39,13, has the same form in the new as in the old coordinate system

$$1 = x^T x = (Uy)^T Uy$$
$$= y^T y$$

When restricted to $x^T x = 1 = y^T y$, the values of the isograms $\{F \equiv x^T A x = \text{const}\}$ have extreme values

$$F = x^T A x = \lambda_1 \quad \text{when } x = \vec{x}_1$$
$$F = x^T A x = \lambda_2 \quad \text{when } x = \vec{x}_2$$

$F\big|_{\|x\|^2=1}$ = extremum on the unit circle here and here