LECTURE 39

I.) THE UNIVERSE, MATHEMATICS AND QUADRATIC FORMS

II.) POSITIVE DEFINITE MATRICES

III.) THREE FACES OF POSITIVE DEFINITENESS

IV.) GEOMETRY OF QUADRATIC FORMS:

A. Vibrating System: Energy Conservation

B. Vibrating System: Normal Modes & Their Energy

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I.) THE UNIVERSE, MATHEMATICS AND QUADRATIC FORMS.

The complexity of mathematics is a reflection of the complexity of the relationships that exist in the universe.

Progress in science and engineering necessarily depends on identifying these relationships and then casting them into numerical, i.e. mathematical, form.

A significant part of these relationship is conceptualized in terms of multi-variable scalar functions $F(x^1, x^2, \ldots, x^n) \equiv F(\vec{x})$ represented in terms of a Taylor

$$F(\vec{x}) = F(\vec{0}) + \frac{\partial F}{\partial x^i} x^i + \frac{1}{2} x^i \overbrace{\boxed{\frac{\partial^2 F}{\partial x^i \partial x^j}}}^{A_{ij}} x^j + \cdots$$

$$\boxed{F(\vec{x}) = F(\vec{0}) + \langle \vec{b}, \vec{x} \rangle + \langle \vec{x}, A\vec{x} \rangle + \cdots}$$

and the linear mathematics surrounding it.

Of particular physical interest is the circumstance where $\vec{x}=\vec{0}$ is a critical point, which is characterized by

$$\frac{\partial F(\vec{0})}{\partial x^i}=0 .$$

In that case the local behaviour of $F(\vec{x})$ in the neighborhood of this point is controlled entirely by the symmetric matrix

$$[A_{ij}] = \left[\frac{1}{2}\frac{\partial^2 F(\vec{0})}{\partial x^i \partial x^j}\right]$$

Thus the study of $F(\vec{x}) = F(\vec{0}) + \langle \vec{x}, A\vec{x}\rangle + \cdots$ reduces to a study of quadratic forms $\langle \vec{x}, A\vec{x}\rangle$.

Side Comment (Q: Need A be symmetric? A: YES!)

In studying a quadratic form

$$\langle x, Ax \rangle = x^i A_{ij} x^j$$

one might ask: Can A be antisymmetric?

The answer is NO. Indeed, consider

$$x^i A_{ij} x^j = \frac{1}{2} x^i (A_{ij} + A_{ji}) x^j + \frac{1}{2} x^i (A_{ij} - A_{ji}) x^j.$$

In the second double sum let $i = j'$ and $j = i'$, and then drop the "prime":

$$x^i A_{ij} x^j = \frac{1}{2} x^i (A_{ij} + A_{ji}) x^j + \frac{1}{2} x^i A_{ij} x^j - \frac{1}{2} x^{j'} A_{i'j'} x^{i'}$$

$$= \quad \text{''} \quad \text{''} \quad - \frac{1}{2} x^{i'} A_{i'j'} x^{j'} \qquad i \leftrightarrow j$$

$$= \quad \text{''} \quad \underbrace{+ \frac{1}{2} x^i A_{ij} x^j - \frac{1}{2} x^i A_{ij} x^j}_{\text{zero}}$$

Thus

$$\boxed{x^i A_{ij} x^j = \frac{1}{2} x^i (A_{ij} + A_{ji}) x^j}$$

One therefore arrives at the following

Conclusion:

(i) Only the symmetric part of A, namely $\frac{1}{2}(A+A^T)$, determines and is determined by the quadratic form.

(ii) The quadratic form $\langle x, Ax \rangle$ implies that $\exists$ a unique symmetric matrix A,

Knowing $\langle x, Ax \rangle \; \forall x \iff$ Knowing $\{A(i,j)\}$; i.e. $\langle x, Ax \rangle \leftrightarrow [A(i,j)]$

(iii) Given $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$,

$\langle x, Ax \rangle$ is determined by $\frac{1}{2}(A+A^T)$;

$\frac{1}{2}(A-A^T)$ contributes nothing.

II.) POSITIVE DEFINITE MATRICES

The function

$$F(\vec{x}) = F(\vec{0}) + \langle \vec{x}, A\vec{x} \rangle$$

has a minimum whenever

$$\boxed{\langle \vec{x}, A\vec{x} \rangle > 0 \qquad \forall\, \vec{x} \neq \vec{0}.} \qquad (*)$$

Definition

Matrix A is said to be *positive definite* whenever condition $(*)$ is satisfied.

Example: $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

Consider

$$\langle \vec{x}, A\vec{x} \rangle = [x\, y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= a x^2 + 2bxy + cy^2 \equiv f$$

By completing the square one obtains

$$\langle \vec{x}, A\vec{x} \rangle = a\left(x + \frac{b}{a} y\right)^2 + \left(c - \frac{b^2}{a}\right) y^2$$

Conclusion:

i) $a > 0$
ii) $\underbrace{ac - b^2}_{\det\begin{vmatrix} a & b \\ b & c \end{vmatrix}} > 0$ $\Big\} \Leftrightarrow$ A is positive definite

or equivalently

i) $a > 0$
ii) $\det\begin{vmatrix} a & b \\ b & c \end{vmatrix} > 0$ $\Big\} \Leftrightarrow$ A is positive definite

Comment

a) A is said to be positive semi-definite
if i) $a > 0$
ii) $ac - b^2 = 0$

In that case $\langle \vec{X}, A\vec{X} \rangle \geq 0$, in particular, $X = -\frac{b}{a} y \Rightarrow \langle X, AX \rangle = 0$

b) A is said to be indefinite if $ac - b^2 < 0$

In that case $\langle \vec{x}, A\vec{x} \rangle \gtreqless 0$ depending on x & y

For example if

(i) $a = c = 0$, then $\langle \vec{x}, A\vec{x} \rangle = 2bxy$
(ii) $a = 1, c = -1, b = 0$, then $\langle \vec{x}, A\vec{x} \rangle = x^2 - y^2$

III. THREE FACES OF POSITIVE DEFINITENESS

The positive definiteness of an $n \times n$ matrix A can be stated in three equivalent ways according to the following

Theorem (Why are positive definite matrices important?)

1. $x^T A x > 0 \quad \forall x$

$\Updownarrow$

2. All eigenvalues of A satisfy $\lambda_i > 0$

$\Updownarrow$

3. $\exists$ a non non-singular matrix W such that

$$A = W^T W$$

Proof: $1. \Rightarrow 2$

Consider each of the e.v. x_i: $A x_i = \lambda_i x_i \quad i = 1, 2, \ldots, n$

$$\text{Then } 0 < x_i^T A x_i = \lambda_i \, x_i^T x_i = \lambda_i \|x_i\|^2 \quad \text{for } i = 1, \ldots, n$$

$2. \Rightarrow 1.$

Let $\{x_i\}$ be an O.N. eigenvector basis

$$\begin{aligned} x^T A x &= (c_1 x_1^T + \cdots + c_n x_n^T) A (c_1 x_1 + \cdots + c_n x_n) \\ &= (c_1 x_1^T + \cdots + c_n x_n^T)(c_1 \lambda_1 x_1 + \cdots + c_n \lambda_n x_n) \\ &= c_1^2 \lambda_1^2 + \cdots + c_n^2 \lambda_n^2 > 0 \end{aligned}$$

2. $\Rightarrow$ 3

$$A = U\Lambda U^T = U\Lambda^{1/2}\Lambda^{1/2}U^T$$

$$= \underbrace{\left(\Lambda^{1/2}U^T\right)^T}_{W^T} \underbrace{\Lambda^{1/2}U^T}_{W}$$

Thus

$$A = W^T W$$

3. $\Rightarrow$ 1.

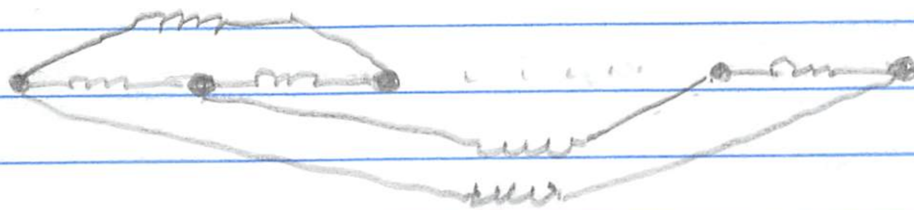
$$A = W^T W \therefore x^T A x = x^T W^T W x$$

$$= (Wx)^T Wx = \|Wx\|^2 > 0 \quad \forall x.$$

IV.) GEOMETRY OF QUADRATIC FORMS:

A. Vibrating System: Energy Conservation

Consider a linear system of n masses interconnected by springs



a) The state of these masses is expressed by their displacement away from equilibrium and their velocities

$$\begin{bmatrix} q^1(t) \\ \vdots \\ q^n(t) \end{bmatrix} \equiv q(t)\,; \quad \frac{d}{dt}\begin{bmatrix} q^1(t) \\ \vdots \\ q^n(t) \end{bmatrix} \equiv \dot{q}(t)$$

b) The multicomponent force of restitution is expressed in terms of the system's Hook's matrix A, which is symmetric:

$$\text{force}(t) = -A\,q(t)\,.$$

c) Newton's eq'ns of motion for n identical unit masses is

$$\ddot{q}(t) + A q(t) = 0 \qquad (*)$$

d) The system is time-invariant. Consequently, its total energy (T.E.) is constant. Indeed, multiplying on the left by the system velocity $\dot{q}^T$ one finds

$$\dot{q}^T \ddot{q} + \dot{q}^T A q = 0$$

for normal mode; to be extremized at a moment of time symmetry ($\dot{q} = 0$)

or

$$\frac{d}{dt}\Big\{ \underbrace{\underbrace{\tfrac{1}{2}(\dot{q}^T \dot{q})}_{\text{K.E.}} + \underbrace{\tfrac{1}{2} q^T A q}_{\text{P.E. = Potential Energy}}}_{\text{T.E}} \Big\} = 0$$

Thus

$$\text{T.E.} = \tfrac{1}{2}\|\dot{q}\|^2 + \tfrac{1}{2} q^T A q = \text{constant} \qquad (**)$$

is a constant of motion, independent of time.

For the time-invariant linear system

$$B\ddot{\vec{q}}(t) = A\vec{q}(t)$$

Consider the i^{th} normal mode

$$\ddot{f}_i + \lambda_i f_i = 0 \qquad f_i = c_i \sin(\omega_i t + \delta_i), \quad \omega_i = \sqrt{\lambda_i}$$

$$A\vec{X}_i = \lambda_i \vec{X}_i$$

where

$$X_i^T X_j = \delta_{ij}$$

Superposition of normal modes

$$q(t) = \sum f_i(t) X_i$$

Total energy = K.E. + P.E. obtained from $\dot{q}^T B \ddot{q} = \dot{q}^T A q$

$$= \frac{d}{dt}\left[\frac{1}{2}\dot{q}^T B \dot{q}^T + \frac{1}{2} q^T A q\right] \text{ is}$$

~~T.E.~~ =

$$= \frac{1}{2}\dot{q}^T \overset{B}{} \dot{q} + \frac{1}{2} q^T A q$$

$$= \frac{1}{2}\sum_i \left(\dot{f}_i X_i\right)^T \sum_j \dot{f}_j \overset{B}{} X_j + \frac{1}{2}\sum\sum f_i f_j \; X_i^T A X_j$$

$$= \frac{1}{2}\sum_i\sum_j \dot{f}_i \dot{f}_j \,\delta_{ij} + \frac{1}{2}\sum\sum f_i f_j \; \underbrace{X_i^T \lambda_j X_j}_{\lambda_j \delta_{ij}}$$

$$= \frac{1}{2}\sum (\dot{f}_i)^2 + \frac{1}{2}\sum \lambda_i (f_i)^2$$

$$= \frac{1}{2}\sum c_i^2 \omega_i^2 \cos^2(\omega_i t + \delta_i) + \frac{1}{2}\sum c_i^2 \omega_i^2 \sin^2(\omega_i t + \delta_i)$$

$$TE = \frac{1}{2}\sum_i \frac{1}{2} c_i^2 \omega_i^2$$

= sum of energies in each normal mode

conserved