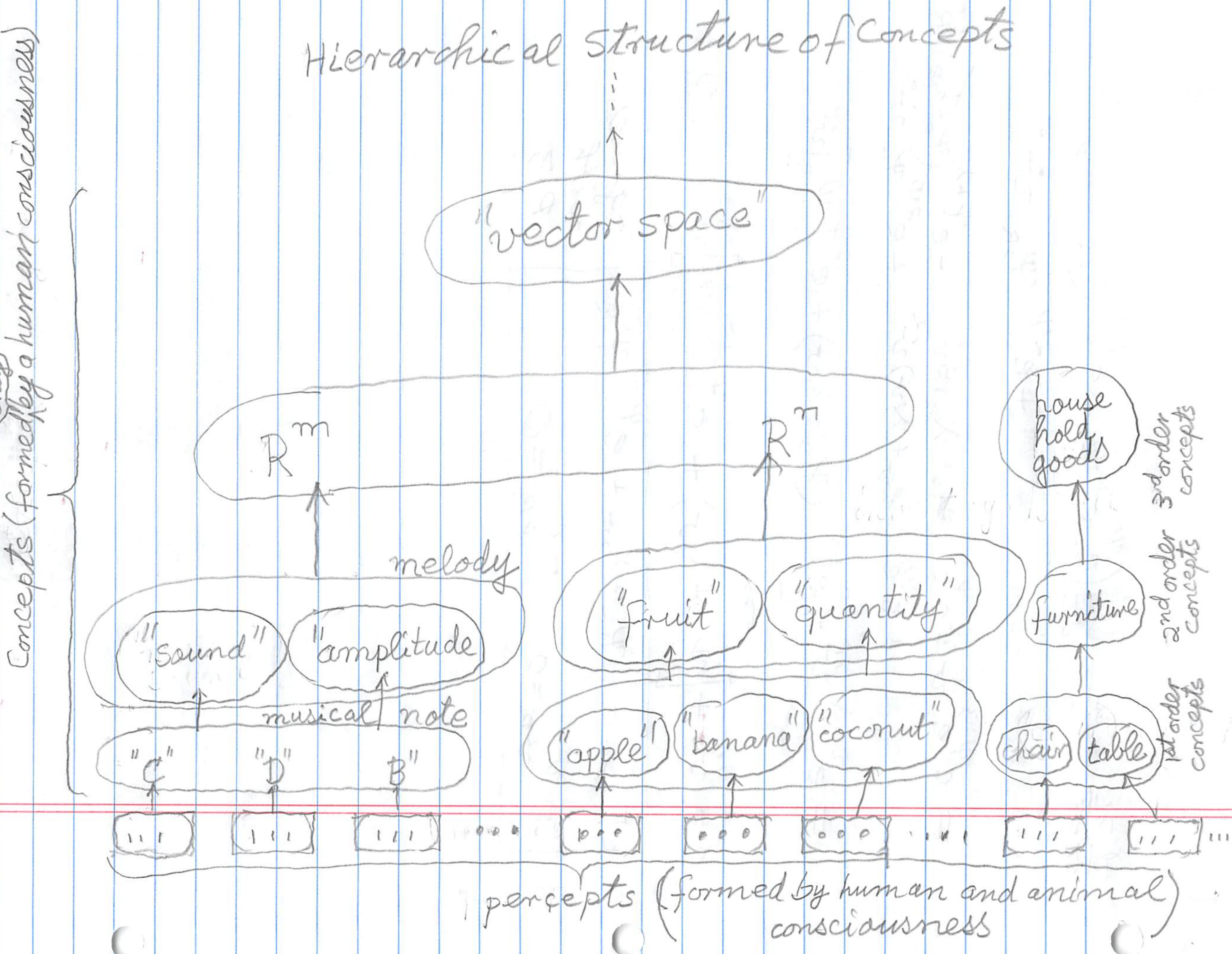
LECTURE 3

1. The Hierarchical Structure of Concepts
2. Spanning sets: examples
3. Linear independence
4. Basis; coordinates

For a subsequent Lecture: Basis-induced isomorphism.

Hierarchical Structure of Concepts



The Hierarchical Nature of Concepts

(An explanation of the diagram on the previous page)

Both knowledge and concepts have a hierarchical structure. Before one can grasp calculus one must have grasped algebra; before grasping algebra one must have grasped arithmetic.

Such a hierarchical relationship also holds for concepts, including those that make up linear algebra.
Q: What is at the base of this hierarchy?
A: Its epistemic starting point is the evidence of the senses, a fact already pointed out by Aristotle some 2400 years ago.

By means of a selective focus and a process of measurement omission (Ayn Rand 1905-1982) one integrates

- two or more instances of, say, concrete apples into the concept "apple",
- two or more instances of concrete bananas into the concept "banana",
- two or more instances of the lemons into the concept "lemon", etc.

Similarly, one integrated two or more instances the musical tones C into the auditory/musical concept "C", two or more instances of musical tones D into the auditory/musical concept "D", etc.

Next, first order concepts such as these get integrated by the same method (selective focus + measurement omission, both by one's faculty of consciousness) into higher order concepts.

Thus, "apple", banana", "lemon", etc. get integrated into the concept "fruit"; then, by combining it with the mathematical concept "quantity", the grocery store manager forms the higher compound concept "fruit inventory".[1]
Meanwhile "C", "D","E", etc. get integrated into the musical concept "note", and then, by combining particular instances of it with the quantitative concepts "time", "duration", and amplitude", into the higher order compound concept "melody" – more generally – an "auditory signal".

The process of integrating concretes into concepts, and concepts into higher order abstractions we owe to the Greeks (and also, more recently, to Ayn Rand). Thales of Miletus (640-550), an engineer, philosopher, mathematician and scientist, opened the door to philosophy, science, and mathematics by being the first man in recorded history to face the bewildering diversity of things by asking: "Do they all something in common? Is there an underlying similarity that unites into an intelligible whole the riot of differences; and if so, what is it?".

The Greeks came to formulate that quest into abstract and immortal terms. What, they asked, is the One[2] in the Many?

Following that trail blazing epistemic lead pioneered by the Greeks, as mathematical engineers and scientists, we ask, "What do auditory signals and fruit inventories have in common? A moment's reflection and comparison with the definition (in Lecture 1) of an abstract vector space leads to the conclusion that these signals and inventories both have all eight of the properties that define an abstract vector space.

One concludes that the One in these audio signals, fruit inventories, and many other examples like these is the concept of an (abstract) vector space. Such a conclusion is a step forward. Indeed, abstractions such as this (i.e. having their basis in the physical world) is what is needed to grasp relations that exist in the universe.

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1 We shall see below that this is an instance of a still higher order concept, a "vector".

2 Following common practice, I am capitalizing One, and Many below, when I use them in the Greek sense.

Reminder (Theorem 2)
Given a spanning set $Q \subset V$,
$Sp(Q)$ is a subspace of V.

Example:
Consider the two spanning sets

$$\left.\begin{array}{l} Q_1 = \{1+x,\ x+x^2,\ 1-x^2\} \\ \\ Q_2 = \{1+x,\ x+x^2,\ 1+x^2\} \end{array}\right\} \subset P_2 = \{p : p(x) = a_0 + a_1 x + a_2 x^2\}$$

$Sp(Q_1)$ and $Sp(Q_2)$ are subspaces of P_2.

Question: Is Q_1 a spanning set for P_2?
Is Q_2 a spanning set for P_2?
Answer: Construct $Sp(Q_1)$ and $Sp(Q_2)$, and observe the difference.

The constructions proceed as follows:

For Q_1 we consider a linear combination determined by

$$(1+x)u+(x+x^2)v+(1-x^2)w=p=\underbrace{a_0+a_1x+a_2x^2}_{\text{given}}$$

1.) Can we solve for u, v, and w in terms of a_0, a_1, a_2? If yes, then Q_1 is a spanning set

2.) If so, what are u, v, and w?

Example 1 (Spanning set for $\mathcal{P}_2$?)

a) Question: Is $Q=\{1+x,\ x+x^2,\ 1-x^2\}$ a spanning set for $\mathcal{P}_2$?

Answer: Let $p = a_0 + a_1 x + a_2 x^2 \in \mathcal{P}_2$.

The question is answered by asking whether

(*) $$(1+x)u + (x+x^2)v + (1-x^2)w = p = a_0 + a_1 x + a_2 x^2$$

can be solved for (u, v, w) for any $p \in \mathcal{P}_2$;
i.e. do there exist (u, v, w) such that Eq. (*) is satisfied.

Method I:
Equating equal powers of x one obtains

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{array}{c} \begin{bmatrix} a_0 \\ a_1 \\ a_3 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow x^0 \\ \leftarrow x^1 \\ \leftarrow x^2 \end{array}$$

To solve this system one applies Gaussian elimination to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a_0 \\ 1 & 1 & 0 & a_1 \\ 0 & 1 & -1 & a_2 \end{array}\right]$$

This is a symbolic way of writing 3 equations in the 3 unknowns u, v, w. They do not occur in this matrix because the to-be-done row operations do not alter the u, v, and w.

The Gaussian elimination yields

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a_0 \\ 0 & 1 & -1 & a_1 - a_0 \\ 0 & 0 & 0 & a_2 - a_1 + a_0 \end{array}\right] \begin{array}{l} \leftarrow u + w = a_0 \\ v - w = a_1 - a_0 \\ 0 \cdot w = a_2 - a_1 + a_0 \end{array}$$

GO TO P 3.4b

Thus Q_1 is spanning set only for those elements of P_2 for which $a_2 = a_1 - a_0$, i.e. only for polynomial whose form is

$$\{ p(x) = a_0 + a_1 x + (a_1 - a_0) x^2 \} = sp(Q_1) \subset V$$

Conclusion: Q_1 is <u>not</u> a spanning set for V because $Sp(Q_1) \neq V$:

$sp(Q_1)$ is a proper subspace of V.

Conclusion:

$$\text{1. } u(1+x)+v(x+x^2)+w(1-x^2)=p(x)= \overset{\text{Given}}{a_0+a_1x+a_2x^2}$$

only when $a_2 = a_1 - a_0$

In that case

$$p = (a_0 - w)(1+x) + (a_1 - a_0 + w)(x+x^2)$$
$$= a_0 + a_1 x + (a_1 - a_0)x^2 + w(x^2 - 1)$$

w = any real number

2. Thus Q_1 is a spanning set only for those elements of $\mathcal{P}_2$ for which $a_2 = a_1 - a_0$

i.e. only for polynomials whose form is

$$\{p(x) = a_0 + a_1 x + (a_1 - a_0)x^2\} = Sp(Q_1) \subset V = \mathcal{P}_2$$

3. a) Q_1 is <u>not</u> a spanning set for $V = \mathcal{P}_2$ because

$$Sp(Q_1) \neq V$$

b) $Sp(Q_1)$ is a proper subspace of $V = \mathcal{P}_2$:

$$Sp(Q_1) \subset \mathcal{P}_2$$

GO TO P 3.6

SKIP This is class

Method II: Evaluate x for three values to obtain u, v, w

$$(1+x)u + (x+x^2)v + (1-x^2)w = p = a_0 + a_1 x + a_2 x^2$$

a) Let $x = 0$ and obtain

$$u + w = a_0$$

Let $x = 1$ and obtain

$$2u + 2v = a_0 + a_1 + a_2 = 2a_1 \rightarrow v = a_1 - u$$

Let $x = -1$ and obtain

$$0 = a_0 - a_1 + a_2 \rightarrow a_2 = -a_0 + a_1$$

b) Solve for

$$u = a_0 - w$$
$$v = a_1 - a_0 + w$$
$$w = w$$

c) Express p in terms of $\{(1+x), (x+x^2), (1-x^2)\}$:

$$p = (1+x)(a_0 - w) + (x+x^2)(a_1 - a_0 + w) + (1-x^2)w$$

$$= (1+x)a_0 + (x+x^2)(a_1 - a_0) + \underbrace{\left[-(1+x) + (x+x^2) + 1 - x^2\right]}_{=0} w$$

i.e the w-terms cancel out, as they must!

Conclusion:

$$\boxed{Sp(Q_1) = Sp(\{1+x, x+x^2\}) = \{p = a_0 + a_1 x + (a_1 - a_0)x^2\}}$$

$$p = a_0 + x(a_0 + a_1 - a_0) + x^2(a_1 - a_0)$$

$= a_0 + a_1 x + (a_1 - a_0)x^2$ as required by the condition

$$a_2 = -a_0 + a_1$$

b) Question: Is $Q_2 = \{1+x, x+x^2, 1+x^2\}$ a spanning set for V?

Answer: Yes.

Indeed, applying Gaussian elimination yields

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a_0 \\ 0 & 1 & -1 & a_1 - a_0 \\ 0 & 0 & 2 & a_2 - a_1 + a_0 \end{array}\right]$$

Reduction to echelon form yields

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}(a_0 + a_1 - a_2) \\ 0 & 1 & 0 & \frac{1}{2}(-a_0 + a_1 + a_2) \\ 0 & 0 & 1 & \frac{1}{2}(a_0 - a_1 + a_2) \end{array}\right] \begin{array}{l} \leftarrow = u \\ \leftarrow = v \\ \leftarrow = w \end{array}$$

Thus any $p = a_0 + a_1 x + a_2 x^2$

$p(x) = (1+x)u + (x+x^2)v + (1+x^2)w \quad (\star)$

can be written as a linear combination of the elements of Q_2, i.e.

$$P_2 = \text{span}(Q_2)$$

i.e Q_2 is a spanning set for P_2 indeed.

III Linearly Independent Set (of vectors):

The concept of Linear Independence.

A set of vectors has two key properties. Besides the spanning property of a set of vectors, there is its other key property, namely its linear independence (or dependence). These concepts are identified by means of the following definitions.

Definition 4 (Linear dependence / independence)

Let V be a vector space

Let $\{v_1, \ldots, v_p\} \subset V$

Consider the equation

$$a_1 v_1 + \cdots + a_p v_p = \vec{0}$$

If there exists a nontrivial solution, i.e. $\exists\, a_1, \ldots, a_p$ not all zero, then one says that $\{v_1, \ldots, v_p\}$ is a linearly dependent set

Put differently

a) the set $\{v_1, \dots, v_p\}$ is said to be <u>linearly dependent</u> whenever

$$a_1 v_1 + \dots + a_p v_p = \vec{0}$$

has a non-trivial solution.

b) we have Def'n 4a

Def'n 4a: $\boxed{\begin{array}{l} a_1, \dots, a_p \text{ not all zero} \\ \text{is a solution to} \\ a_1 v_1 + \dots + a_p v_p = \vec{0} \end{array}}$ $\Leftrightarrow$ $\{v_1, \dots, v_p\}$ is a <u>lin. dep. set</u>

Stated still more differently, we have Def'n 4b

c) $\{v_1, \dots, v_p\}$ is a <u>linearly independent set</u> if it is <u>not</u> linearly dependent, i.e. we have

Def'n 4b $\boxed{\begin{array}{l} a_1 = \dots = a_p = 0 \text{ is the only} \\ \text{solution to} \\ a_1 v_1 + \dots + a_p v_p = \vec{0} \end{array}}$ $\Leftrightarrow$ $\{v_1, \dots, v_p\}$ is a <u>linearly indep set.</u>

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Intermediate Summary:

So far we have formed, defined, and related the following concepts:

Vector space	Def. 1
Subspace	Def 2, Thm 1
Q = spanning set Sp(Q) = span of Q	Def. 3a
Sp(Q) is a subspace	Thm 2
Spanning set for V:	Def 3b
Sp(Q) = V	Thm 3
Linearly dependent set	Def. 4a
Linearly independent set	Def. 4b

IV. Basis For a Vectorspace

By applying the concept of linear independence to a spanning set, say Q, one arrives at the concept of a <u>basis</u> for $Sp(Q)$.

Example 2

Again, consider $Q_2 = \{1+x,\ x+x^2,\ 1+x^2\}$.

and Eq. (★) on page 3.6 with $a_0 = a_1 = a_2 = 0$,

$(1+x)u + (x+x^2)v + (1+x^2)w = \vec{0}$ (= zero polynomial)

The result of the reduction to echelon form yields

$$u = 0$$
$$v = 0$$
$$w = 0,$$

and this is the <u>only</u> solution. Consequently, Q is a <u>spanning set</u> (for P_2) which is <u>linearly independent</u>.

By contrast for

$$Q_1 = \{1+x, x+x^2, 1-x^2\}$$

we have

$$(1+x)u + (x+x^2)v + (1-x^2)w = 0 \quad (\forall x)$$

whenever

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} w \quad \text{not all zero;}$$

A linearly independent spanning set, such as Q, is called a basis for P_2.

Thus we have the following

Definition 5a (Basis for a vector space)

(i) Let V = vector space

(ii) Let $B = \{v_1, \ldots, v_p\} \subset V$ be a spanning set for V, i.e.

(iii) $V = Sp(B)$ ("B is a spanning set for V")

If B is linearly independent then B is called a basis. In other words, a linearly independent spanning set is a basis (for the spanned space).