LECTURE 3

1. The Hierarchical Structure of Concepts
2. Spanning sets; examples
3. Linear independence
4. Basis; coordinates

For a subsequent Lecture

Basis-induced isomorphism.
Hierarchical structure of concepts

Concepts (formed by a human consciousness)

Consciousness

Percepts (formed by human and animal)

Concepts

3rd order concepts

2nd order concepts

1st order concepts

Food, goods, furniture

Quantity

Fruit

Sound

Melody

Musical note

Amplitude

Sound

Vector space
The Hierarchical Nature of Concepts
(An explanation of the diagram on the previous page)
Both knowledge and concepts have a hierarchical structure. Before one can grasp calculus one must have grasped algebra; before grasping algebra one must have grasped arithmetic.

Such a hierarchical relationship also holds for concepts, including those that make up linear algebra.
Q: What is at the base of this hierarchy?
A: Its epistemic starting point is the evidence of the senses, a fact already pointed out by Aristotle some 2400 years ago.

By means of a selective focus and a process of measurement omission (Ayn Rand 1905-1982) one integrates
- two or more instances of, say, concrete apples into the concept “apple”;
- two or more instances of concrete bananas into the concept “banana”;
- two or more instances of the lemons into the concept “lemon”, etc.
Similarly, one integrated two or more instances the musical tones C into the auditory/musical concept “C”, two or more instances of musical tones D into the auditory/musical concept “D”, etc.

Next, first order concepts such as these get integrated by the same method (selective focus + measurement omission, both by one’s faculty of consciousness) into higher order concepts.

Thus, “apple”, banana”, “lemon”, etc. get integrated into the concept “fruit”; then, by combining it with the mathematical concept “quantity”, the grocery store manager forms the higher compound concept “fruit inventory”.
Meanwhile “C”, “D”, “E”, etc. get integrated into the musical concept “note”, and then, by combining particular instances of it with the quantitative concepts “time”, “duration”, and amplitude”, into the higher order compound concept “melody” – more generally – an “auditory signal”.

The process of integrating concretes into concepts, and concepts into higher order abstractions we owe to the Greeks (and also, more recently, to Ayn Rand). Thales of Miletus (640-550), an engineer, philosopher, mathematician and scientist, opened the door to philosophy, science, and mathematics by being the first man in recorded history to face the bewildering diversity of things by asking: “Do they all something in common? Is there an underlying similarity that unites into an intelligible whole the riot of differences; and if so, what is it?”

The Greeks came to formulate that quest into abstract and immortal terms. What, they asked, is the One in the Many?

Following that trail blazin epistemic lead pioneered by the Greeks, as mathematical engineers and scientists, we ask, ”What do auditory signals and fruit inventories have in common? A moment’s reflection and comparison with the definition (in Lecture 1) of an abstract vector space leads to the conclusion that these signals and inventories both have all eight of the properties that define an abstract vector space.

One concludes that the One in these audio signals, fruit inventories, and many other examples like these is the concept of an (abstract) vector space. Such a conclusion is a step forward. Indeed, abstractions such as this (i.e. having their basis in the physical world) is what is needed to grasp relations that exist in the universe.

1 We shall see below that this is an instance of a still higher order concept, a “vector”.
2 Following common practice, I am capitalizing One, and Many below, when I use them in the Greek sense.
Reminder (Theorem 2)
Given a spanning set \( Q = V \),
\( \text{Sp}(Q) \) is a subspace of \( V \).

Example:
Consider the two spanning sets
\[
Q_1 = \{1 + x, x + x^2, 1 - x^2\} \\
Q_2 = \{1 + x, x + x^2, 1 + x^2\}
\]
\( \text{Sp}(Q_1) \) and \( \text{Sp}(Q_2) \) are subspaces of \( P_2 \).

Question: Is \( Q_1 \) a spanning set for \( P_2 \)?
Is \( Q_2 \) a spanning set for \( P_2 \)?
Answer: Construct \( \text{Sp}(Q_1) \) and \( \text{Sp}(Q_2) \), and observe the difference.

The constructions proceed as follows:
For Q, we consider a linear combination determined by

\[(1+x)u + (x+x^2)v + (1-x^2)w = p = a_0 + a_1 x + a_2 x^2 \]

given

1) Can we solve for \(u, v, \) and \(w\) in terms of \(a_0, a_1, a_2\)? If yes, then \(Q\) is a spanning set.

2) If so, what are \(u, v, \) and \(w?\)
Example 1 (Spanning set for \( \mathbb{P}_2 \))

a) Question: Is \( Q = \{1+x, x+x^2, 1-x^2\} \) a spanning set for \( \mathbb{P}_2 \)?

Answer: Let \( p = a_0 + a_1 x + a_2 x^2 \in \mathbb{P}_2 \)

The question is answered by asking whether

\[
(\star) \quad (1+x)u + (x+x^2)v + (1-x^2)w = p = a_0 + a_1 x + a_2 x^2
\]

can be solved for \((u, v, w)\) for any \( p \in \mathbb{P}_2 \), i.e. do there exist \((u, v, w)\) such that Eq. \((\star)\) is satisfied?

Method I:

Equate equal powers of \( x \) one obtains

\[
\begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
= 
\begin{bmatrix}
a_0 \\
a_1 \\
a_2
\end{bmatrix} - x^0
\]

To solve this system one applies Gaussian elimination to the augmented matrix

\[
\begin{bmatrix}
1 & 0 & 1 & a_0 \\
1 & 1 & 0 & a_1 \\
0 & 1 & -1 & a_2
\end{bmatrix}
\]
This is a symbolic way of writing 3 equations in the 3 unknowns \( u, v, w \). They do not occur in this matrix because the to-be-done row operations do not alter the \( u, v \), and \( w \).

The Gaussian elimination yields:

\[
\begin{bmatrix}
1 & 0 & 1 & | a_0 \\
0 & 1 & -1 & | a_1 - a_0 \\
0 & 0 & 0 & | a_2 - a_1 + a_0
\end{bmatrix}
\]

\[\begin{align*}
-u + w &= a_0 \\
v - w &= a_1 - a_0 \\
o w &= a_2 - a_1 + a_0
\end{align*}\]

Thus, \( Q_1 \) is spanning set only for those elements of \( \mathbb{B} \) for which

\[a_2 = a_1 - a_0,\]

i.e. only for polynomial whose form is

\[\{ p(x) = a_0 + a_1 x + (a_1 - a_0) x^2 \} = \mathsf{sp}(Q_1) \subset V\]

Conclusion: \( Q_1 \) is not a spanning set for \( V \) because \( \mathsf{sp}(Q) \neq V \); \( \mathsf{sp}(Q_1) \) is a proper subspace of \( V \).
Conclusion:

1. \( w (1+x) + v (x^2 + x) + w (1-x^2) = p(w) = a_0 + a_1 x + a_2 x^2 \)

   only when \( a_2 = a_1 - a_0 \)

In that case

\[
p = (a_0 - w)(1+ x) + (a_1 - a_0 + w)(x^2 + x) \]
\[
= a_0 + a_1 x + (a_1 - a_0) x^2 + w(x^2 - 1) \quad \text{is a real number}
\]

2. Thus \( Q_1 \) is a spanning set only for those elements of \( P_2 \) for which \( a_2 = a_1 - a_0 \).

   i.e. only for polynomials whose form is

\[
\{ p(w) = a_0 + a_1 x + (a_1 - a_0) x^2 \} = \text{sp}(Q_1) \subset V = P_2
\]

3. a) \( Q_1 \) is not a spanning set for \( V = P_2 \)

   because \( \text{sp}(Q_1) \neq V \)

   b) \( \text{sp}(Q_1) \) is a proper subspace of \( V = P_2 \):

\( \text{sp}(Q_1) \subset P_2 \)
Method II: Evaluate $x$ for three values to obtain $u, v, w$

\[(1+x)u + (x+x^2)v + (1-x^2)w = p = a_0 + a_1 x + a_2 x^2\]

g) Let $x = 0$ and obtain \[u + w = a_0\]

Let $x = 1$ and obtain \[2u + 2v = a_0 + a_1 + a_2 \implies 2a_1 \rightarrow v = a_1 - u\]

Let $x = -1$ and obtain \[0 = a_0 - a_1 + a_2 \rightarrow a_2 = -a_0 + a_1\]

b) Solve for \[u = a_0 - w\]

\[v = a_1 - a_0 + w\]

\[w = \frac{w}{2}\]

c) Express $p$ in terms of $\{1+x, x+x^2, (1-x^2)\}$:

\[p = (1+x)(a_0-w) + (x+x^2)(a_1-a_0+w) + (1-x^2)w\]

\[= (1+x)a_0 + (x+x^2)(a_1-a_0) + \frac{1}{2}(1+x)(x+x^2) + 1-x^2\frac{w}{2}\]

i.e. the $w$-terms cancel out.

Conclusion:

\[\text{Sp}(A_1) = \text{Sp}(\{1+x, x+x^2\}) = \{p = a_0 + a_1 x + (a_1-a_0)x^2\}\]

\[p = a_0 + x(a_0 + a_1 - a_0) + x^2(a_1 - a_0)\]

\[= a_0 + a_1 x + (a_1 - a_0)x^2 \text{ as required by the condition}\]

\[a_2 = -a_0 + a_1\]
b) Question: Is \(Q_2 = \{1, x, 1 + x, x^2\} \) a spanning set for \(V_2\)?

Answer: Yes.

Indeed, applying Gaussian elimination yields

\[
\begin{bmatrix}
1 & 0 & 1 & | & a_0 \\
0 & 1 & -1 & | & a_1 - a_0 \\
0 & 0 & 2 & | & a_2 - a_1 + a_0
\end{bmatrix}
\]

Reduction to echelon form yields

\[
\begin{bmatrix}
1 & 0 & 0 & | & \frac{1}{2} (a_0 + a_1 - a_2) \\
0 & 1 & 0 & | & \frac{1}{2} (-a_0 + a_1 + a_2) \\
0 & 0 & 1 & | & \frac{1}{2} (a_0 - a_1 + a_2)
\end{bmatrix}
\]

Thus any \( p = a_0 + a_1 x + a_2 x^2 \)

\[ p(x) = (1 + x)u + (x + x^2)v + (1 + x^2)w \] \((\star)\)

can be written as a linear combination of the elements of \(Q_2\), i.e.

\[ p_0 = \text{span} \{Q_2\} \]

i.e. \(Q_2\) is a spanning set for \(V_2\) indeed.
Linearly Independent Set (of vectors):

The concept of linear independence:
A set of vectors has two key properties. Besides the spanning property of a set of vectors, there is its other key property, namely its linear independence (or dependence). These concepts are identified by means of the following definitions.

Definition 4 (Linear dependence/independence)

Let $V$ be a vector space

Let $\mathbf{v}_1, \ldots, \mathbf{v}_p \subseteq V$

Consider the equation

$$a_1 \mathbf{v}_1 + \cdots + a_p \mathbf{v}_p = \mathbf{0}$$

If there exists a non-trivial solution, i.e. $\exists a_1, \ldots, a_p$ not all zero, then one says that $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is a linearly dependent set.
Put differently

1) The set \( \{v_1, \ldots, v_p\} \) is said to be **linearly dependent** whenever

\[
a_1 v_1 + \cdots + a_p v_p = 0
\]

has a non-trivial solution.

b) **We have Defn 4a**

- **Defn 4a**: \( a_1, \ldots, a_p \) not all zero

is a solution to

\[
a_1 v_1 + \cdots + a_p v_p = 0
\]

iff \( \{v_1, \ldots, v_p\} \) is a limit dependent set.

Stated still more differently, we have Defn 4b

c) \( \{v_1, \ldots, v_p\} \) is a linearly independent set if it is not linearly dependent,

i.e., we have

- **Defn 4b**: \( a_1 = \cdots = a_p = 0 \) is the only solution to

\[
a_1 v_1 + \cdots + a_p v_p = 0
\]

iff \( \{v_1, \ldots, v_p\} \) is a linearly independent set.
Intermediate Summary:

So far we have formed, defined, and related the following concepts:

- **Vector space**  Def. 1
- **Subspace**  Def 2, Thm 1
  - \( Q = \text{spanning set} \)
  - \( \text{Sp}(Q) = \text{span of } Q \)
  - \( \text{Sp}(Q) \) is a subspace
  - \( \text{Spanning set for } V \)
  - \( \text{Sp}(Q) = V \)
  - Thm 2
  - Def 3a
  - Thm 3
  - Def 3b

- **Linearly dependent set**  Def. 4a
- **Linearly independent set**  Def. 4b
IV. Basis For a Vector space

By applying the concept of linear independence to a spanning set, say $A$, one arrives at the concept of a basis for $Sp(A)$.

Example 2

Again, consider $P_2 = \{1 + x, x + x^2, 1 + x^2\}$.

and Eq. (4) on page 3.6 with $a_0 = 0, a_1 = 0, a_2 = 0$.

$(1 + x)u + (x + x^2)v + (1 + x^2)w = 0^2$ (= zero polynomial)

The result of the reduction to echelon form yields $u = 0$, $v = 0$, $w = 0$, and this is the only solution. Consequently, $P_2$ is a spanning set (for $P_2$) which is linearly independent.
By contrast for

\[ q_1 = \left\{ 1 + x, x + x^2, 1 - x^2 \right\} \]

we have

\[(1 + x)u + (x + x^2)v + (1 - x^2)w = 0 \quad (\forall x)\]

whenever

\[
\begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix} =
\begin{bmatrix}
  -1 \\
  1 \\
  1
\end{bmatrix} \quad w \quad \text{not all zero.}
A linearly independent spanning set such as \( \mathbb{Q} \), is called a basis for \( \mathbb{R} \). Thus we have the following:

**Definition 5a (Basis for a vector space)**

(i) Let \( V \) = vector space

(ii) Let \( B = \{ v_1, \ldots, v_n \} \subseteq V \) be a spanning set for \( V \), i.e.,

\[ V = \text{Sp}(B) \quad ("B\ is\ a\ spanning\ set\ for\ V") \]

If \( B \) is linearly independent then \( B \) is called a basis. In other words, a linearly independent spanning set is a basis (for the spanned space).