Wednesday

# LECTURE 4

Coordinte Representative of a vector relative to a given basis
(Existence & uniqueness)

Coordinates

Without numbers (i.e. scalars) the theory of vector spaces would be a mere aggregate of floating abstractions disconnected and detached from the world

Numbers are the means of quantifying and processing the evidence of the senses.

A given vector is subjected to a measuring process, i.e. by relating it a chosen standard, a chosen basis. The result of the measurement is expressed in terms of numbers relative to the basis vectors, which comprise the basis. The n measurements applied to each one of an aggregate



of vectors get conceptualized (by omitting reference to any particular

Example 1

What is a basis for each of the three vectorspaces

a) $V_1 = \left\{ \begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} \right\}$; b) $V_2 = \{a + bx - bx^2 + ax^3\}$; c) $V_3 = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \right\}$

where a and b are scalars.

a) $\begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{v_1} + b \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}}_{v_2} \equiv a v_1 + b v_2$

$B_1 = \{ v_1, v_2 \}$ is a lin. indep. spanning set for $V_1$

b) $a + b - bx + ax^3 = a(1 + x^3) + b(1 - x)$

$B_2 = \{1 + x^3, 1 - x\}$ is a lin. indep. spanning set for $V_2$.

c) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$B_3 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ is a lin. indep. spanning set for $V_3$



## V The Coordinate Representative of a Vector Relative to a Given Basis

Given a vector $p = a_0 + a_1 x + a_2 x^2$, the application of the linear independence of Q on page 3.3 to Eq.(*)

$$(1+x)u + (x+x^2)v + (1+x^2)w = a_0 + a_1 x + a_2 x^2$$

lead to a very important result, namely that the solution on p 3.3

$$\boxed{\begin{aligned} u &= \tfrac{1}{2}(a_0 + a_1 - a_2) \\ v &= \tfrac{1}{2}(-a_0 + a_1 + a_2) \\ w &= \tfrac{1}{2}(a_0 - a_1 + a_2) \end{aligned}}$$

is unique.

The generalization of this result to an arbitrary vector space is expressed by the following

Theorem 4 (Vector uniquely represented in terms of a basis)

Let $B = \{v_1, \ldots, v_p\}$ be a basis for the vector space $V$.

Let $w \in V$ be any given vector in $V$.

Conclusion:

$w$ has a unique representation relative to $B$, i.e. there exist unique scalars $a_1, \ldots, a_p$ such that $w = a_1 v_1 + \cdots + a_p v_p$

Comment 2: This theorem makes two strong claims, namely existence and uniqueness.

Comment 1: This theorem can be restated as follows:
For every vector $w$ in $V$, $\underbrace{\text{there is}}_{\text{"existence"}}$ $\underbrace{\text{exactly one}}_{\text{"uniqueness"}}$ way to write $w$ as a lin. comb'n of the basis vectors in $B$.



Comment 1:
This theorem can be restated as follows:

For every vector $w$ in $W$, $\underbrace{\text{there is}}_{\text{"exists"}}$ $\underbrace{\text{exactly one}}_{\text{"unique"}}$ way to write $w$ as a linear combination of the basis vectors in $B$.

Comment 2:

This theorem makes two claims, namely existence and uniqueness.

Proof of existence:

B is a spanning set for $V \Rightarrow$ for $w \in V$ $\exists$ scalars $a_1, \dots, a_p$ such that

$$\boxed{w = a_1 v_1 + \dots + a_p v_p} \qquad (1)$$

Proof of uniqueness:

Let $b_1, \dots, b_p$ be another set of scalars with the property that

$$w = b_1 v_1 + \dots + b_p v_p$$

Subtracting one finds

$$w - w = 0 = (a_1 - b_1) v_1 + \dots + (a_p - b_p) v_p$$

B is a basis $\Rightarrow$ B is a linearly independent set

$$\Rightarrow \quad a_1 - b_1 = 0, \dots, a_p - b_p = 0$$

i.e. $a_i = b_i$ for $i = 1, \dots, p$.

Thus Eq. (1) at the top of 3.11, namely,

$$w = \sum_{i=1}^{p} a_i v_i$$

is a unique representation of $w$ indeed.

NOTE TO LECTURER: (Needs elaboration) These scalars comprising the connecting link between abstract linear algebra and the evidence of the senses.



The importance of Theorem 4 is that any vector $w \in V$ determines and is determined by $p$ scalars

$$a_1, \dots, a_p$$

once a basis $B = \{v_1, \dots, v_p\}$ is given (or has been chosen). INSERT on P4.6b This set of a scalars is a new concept, namely, the coordinate representative with respect to a basis. Explicitly, one has the following

Definition 5 b (Coordinates of a vector)

The scalars $a_1, \dots, a_p$ are called the coordinates of $w$ relative to

$$B = \{v_1, \dots, v_p\} \subset V$$

INSERT for p 4.6

There is a 1-1 between vector $w$ in $V$ and $p$-tuples $(a_1, \dots, a_p)$ in $R^p$.

$\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}_B$ vs $w$!



In mathematical notation one writes them as the column array

$$[w]_B \equiv \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}_B ,$$

the <u>coordinate representative</u> of $w$ <u>relative to $B$</u>. The <u>entries</u> of the column array <u>are</u> the <u>coordinates</u> of $w$ <u>relative to $B$</u>.

One needs to emphasize that the column vector $\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}_B \in R^p$ is quite <u>distinct</u> from the vector $w \in V$ (although, as we shall find on page 4.13, they correspond to one another once a basis has been specified). This fact is illustrated by the following



Example 2

a) Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\} = M_{22}$, the vector space of $2\times 2$ matrices.

b) Let $B = \left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

be the <u>standard</u> basis for $V$

c) Let $C = \left\{ F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, F_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, F_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, F_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

be another basis for $V$

d) Let $w = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a known element of $V$

<u>FIND</u> the coordinate representative of $w$ (i) relative to $B$ and (ii) relative to $C$.

Finding these representatives is 3 step process:

Step 1:
Expand $w$ in terms of the basis elements.

(i) Relative to $B$ one has by inspection

$$w = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a E_{11} + b E_{12} + c E_{21} + d E_{22}$$

(ii) Relative to $C$ one sets

$$r \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + s \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + u \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} (= w)$$

and solves for $r, s, t, u$ and obtains

$$\begin{aligned} r + s + t + u &= a \\ s + t + u &= b \\ t + u &= c \\ u &= d \end{aligned}$$

or

$$\begin{aligned} u &= d \\ t &= c - d \\ s &= b - c \\ r &= a - b + c \end{aligned}$$



Step 2
Read out the two coordinate representatives

$$[w]_B = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_B \quad \text{(relative to B)}$$

$$[w]_C = \begin{bmatrix} a-b+c \\ b-c \\ c-d \\ d \end{bmatrix}_C \quad \text{(relative to C)}$$

Comment 1:

The key idea is that even though one has coordinate 4-tuples in both cases, the respective coordinates have entirely different meaning because they refer to entirely different bases.

From the perspective of the relationship of $w$ to physical reality, a chosen basis is the standard relative to which one measures $w$; which is, to say, the coordinate components of $w$ are the result of this measurement. Thus, if one changes the standard of measurement, then the result of the measurement will necessarily also change.



Isomorphism Between $V$ and $R^p$

Given a basis $B = \{v_1, \ldots, v_p\} \subset V$ for vector space $V$ one has

$$w = a_1 v_1 + \cdots + a_p v_p$$

with uniquely defined scalars $\{a_1, \ldots, a_p\}$.

Thus one has

$$w \rightsquigarrow \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} \equiv [w]_p$$

$$V \longrightarrow R^p$$

Conversely,

$$\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} \rightsquigarrow a_1 v_1 + \cdots + a_p v_p \equiv w$$

$$R^p \longrightarrow V$$

Thus one has the following

Deferred to Lect. 5 4.13

**Proposition** ( $V \longleftrightarrow R^p$,

a) A basis $B = \{v_1, \dots, v_p\} \subset V$ for vector space $V$

induces a 1-1 mapping

$$V \longleftrightarrow R^p$$

$$w \leftrightsquigarrow [w]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}$$

b) This mapping preserves structure,

namely
(i) addition
(ii) scalar multiplication
(iii) linear combinations
(iv) linear independence

i.e. one has

$$\left.\begin{matrix}(i)\\(ii)\\(iii)\end{matrix}\right\} \quad [c_1 w_1 + c_2 w_2]_B = c_1 [w_1]_B + c_2 [w_2]_B$$

(iv)} $\{w_1, \dots, w_k\}$ is linearly independent (dependent)

$\Leftrightarrow \{[w_1]_B, \dots, [w_k]_B\}$ is lin. indep. (depend.)