LECTURE 40

I, EXTEMUM PRINCIPLE FOR NORMAL MODES

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IV THE EXREMUM PRINCIPLE GEOMETRIZED

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I Extremum Principle for Normal Modes

It is a consequence of Newton's law of motion, Eq. (*) on page 39.10, applied to a normal mode that its amplitude profile x, which obeys

$$Ax = \lambda x,$$

is determined by an extremum principle.

At a moment of time symmetry of a normal mode one minimizes/extremizes the potential (= total) energy of the system under the constraint that it be in a a non-trivial state.

More precisely, one has the following

Proposition

The extremum problem

$$F(x_1, \ldots, x_n) \equiv x^T A x = \underline{\text{extremum}} \quad (1)$$

subject to the constraint

$$x^T x = 1 \quad (2)$$

gives rise to the eigenvalue problem

$$A x = \lambda X .$$

Comment

1. Because of the constraint, the extremum principle is implemented with a Lagrange multiplier, say λ:

$$\frac{\partial}{\partial x^k}\left[x^T A X - \lambda(x^T x - 1)\right] = 0 \qquad k = 1, \dots, n$$

or

$$\nabla\left[\frac{1}{2}(X^T A X)\right] = \lambda X$$

or

$$A X = \lambda X .$$

2. This extremum principle is the bridge between the geometry of the quadratic form $x^T A X$ and the algebra of the eigenvalue problem

$$A x = \lambda X ,$$

II. Geometrization via Concentric Ellipsoids.

The geometrization of $Ax = \lambda x$ is achieved by introducing into the quadratic form $F(x^1, \dots, x^n)$ those variables

$$y = \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix}$$

which are related to the old variables $x = \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix}$ by

$$x = Uy.$$

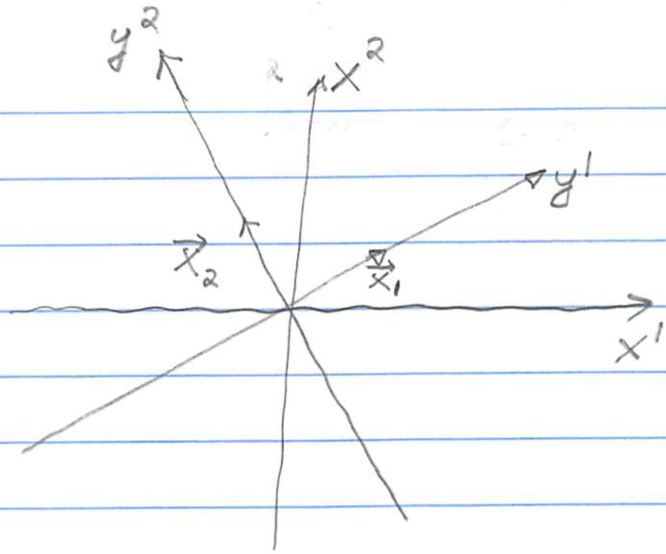
Here

$$U = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}$$

is the unitary transformation which diagonalizes A ($\Lambda = U^{-1}AU$).

This means that the new coordinate axes point along the eigenvectors of A.

In two dimensions one has



$$\begin{bmatrix} y^1 \\ y^2 \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{x}_1 & \vec{x}_2 \\ | & | \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$$

eigenvectors of A

Introducing the new coordinates $\{y^1, \ldots, y^n\}$ into the quadratic form Eq. (1) on page 40.1 one finds.

$$F = x^T A x = y^T U^T A U y.$$

$$= y^T \Lambda y = \lambda_1 (y^1)^2 + \lambda_2 (y^2)^2 + \cdots + \lambda_n (y^n)^2$$

$$= \frac{(y^1)^2}{\frac{1}{\lambda_1}} + \frac{(y^2)^2}{\frac{1}{\lambda_2}} + \cdots + \frac{(y^n)^2}{\frac{1}{\lambda_n}}$$

For a <u>positive definite</u> maxtrix A, one has from page 39.7

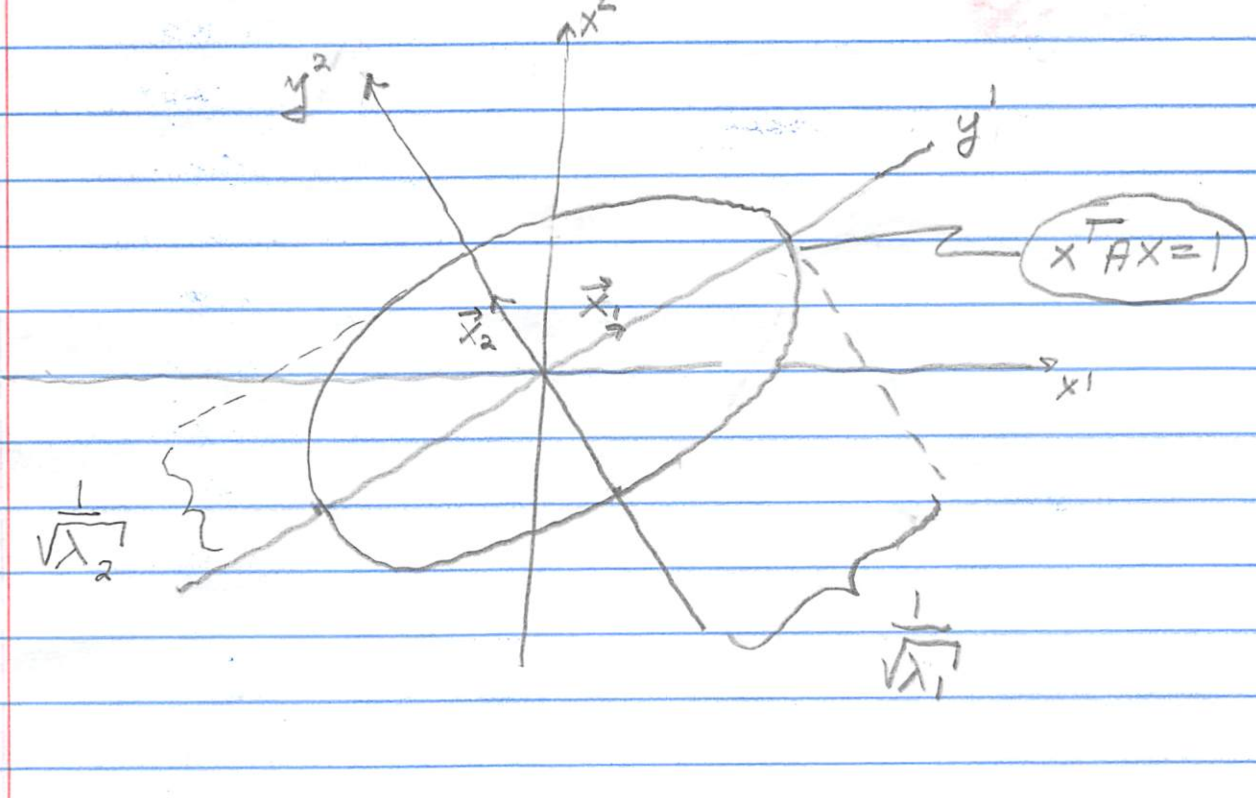
$$0 < \lambda_1 < \cdots < \lambda_n$$

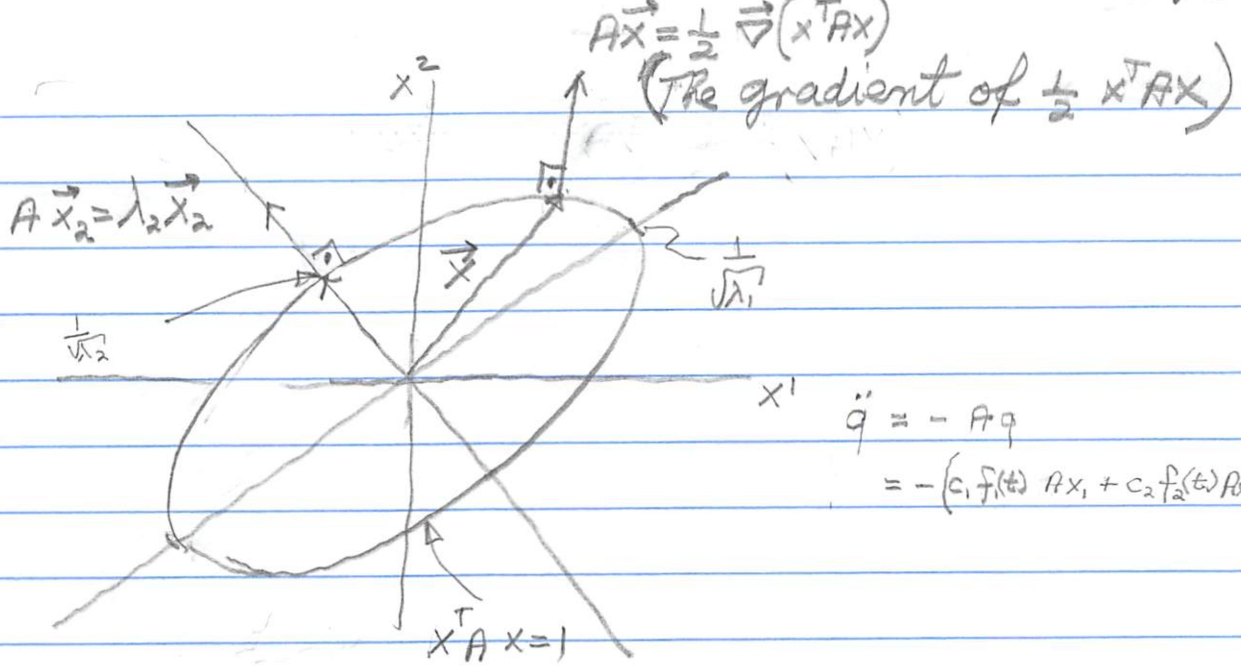
(all positive eigenvalues)

Thus the isograms $F = \text{const}$ are concentric ellipsoids whose axes point along the direction of the eigenvectors $\vec{x}_1, \ldots, \vec{x}_n$.

In particular the $F = 1$ ellipsoid in 2-D is

$$x^T A x = \frac{(y^1)^2}{\left(\frac{1}{\sqrt{\lambda_1}}\right)^2} + \frac{(y^2)^2}{\left(\frac{1}{\sqrt{\lambda_2}}\right)^2} = 1$$





$$\ddot{q} = -Aq$$
$$= -\left(c_1 f_1(t)\, Ax_1 + c_2 f_2(t)\, Ax_2\right)$$

The 1-1 correspondence between a symmetric matrix A and its quadratic form $x^T A x$ leads to the following conclusion:

A symmetric matrix with positive eigenvalues should be pictured as an ellipsoid ① whose semimajor axes point along the eigenvectors of the eigenvector matrix

$$U = \begin{bmatrix} \vdots & & \vdots \\ x_1 & \cdots & x_n \\ \vdots & & \vdots \end{bmatrix}$$

and

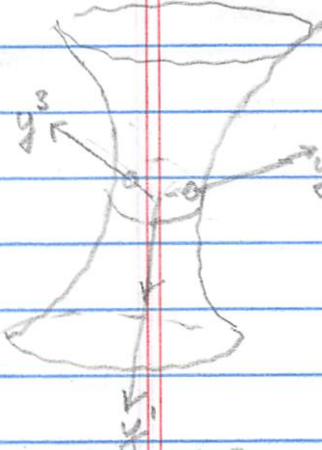
② whose

$$(i^{th} \text{ semi-major axis})^2 = \frac{1}{\lambda_i}$$

III Geometrization via Concentric Hyperboloids In 3 dimensions

If a) $0 < \lambda_1 < \lambda_2 < \lambda_3$ $\quad F = 1$ an ellipsoid

If b) $\lambda_1 < 0 < \lambda_2 < \lambda_3$



$$F = \frac{(y^3)^2}{\frac{1}{\lambda_3}} + \frac{(y^2)^2}{\frac{1}{\lambda_2}} - \frac{(y^1)^2}{\frac{1}{|\lambda_1|}} = 1$$

a hyperboloid of one sheet

If c) $\lambda_1 < \lambda_2 < 0 < \lambda_3$

$$F = \frac{(y^3)^2}{\frac{1}{\lambda_3}} - \frac{(y^2)^2}{\frac{1}{|\lambda_2|}} - \frac{(y^1)^2}{\frac{1}{|\lambda_1|}} = 1$$

a hyperboloid of two sheet

IV. The Extremum Principle Geometrized.

The constraint, Eq. (2) on page 39.13, has the same form in the new as in the old coordinate system

$$1 = x^T x = (Uy)^T Uy$$
$$= y^T y$$

When restricted to $x^T x = 1 = y^T y$, the values of the isograms $\{F \equiv x^T A x = \text{const}\}$ have <u>extreme values</u>

$$F = x^T A x = \lambda_1 \quad \text{when } x = \vec{x}_1$$
$$F = x^T A x = \lambda_2 \quad \text{when } x = \vec{x}_2$$

$F\big|_{\|x\|^2=1}$ = extremum on the unit circle <u>here</u> and <u>here</u>