LECTURE 41

I. Simultaneous Diagonalization of Two Quadratic Forms

A. General Coupled System

B. Normal Modes

C. Eigenvalue Problem

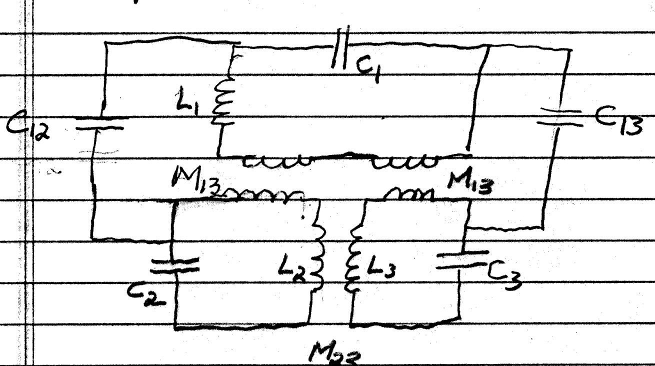
(1) Eigenvalue equation
(2) Eigenvectors
(3) B-orthonormality

D. Change of Variables for Diagonalization.

I. SIMULTANEOUS DIAGONALIZATION OF TWO QUADRATIC FORMS.

A) General Coupled System:

Consider the equation of motion for the following system of inductively coupled networks



The equations of motion for this system are

$$\begin{bmatrix} L_1 & M_{12} & M_{13} \\ M_{12} & L_2 & M_{23} \\ M_{13} & M_{23} & L_3 \end{bmatrix} \frac{d^2}{dt^2} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1} & \frac{1}{C_{12}} & \frac{1}{C_{13}} \\ \frac{1}{C_{12}} & \frac{1}{C_2} & 0 \\ \frac{1}{C_{13}} & 0 & \frac{1}{C_3} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = 0$$

Here $\{L_i\}$ are the self-inductances

$\{M_{ij} = M_{ji}\}$ are the mutual inductances

$\{C_i\}$ are capacitances

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$\{I_i\}$ are the currents.

More generally, we consider the following generalization of Eq.(*) on P39.10

$$B\ddot{q} + Aq = 0.$$

Here $A^T = A$,

$B^T = B$,

$q^T B q > 0 \quad \forall$ vectors q.

That is B is a positive definite matrix.

B) Normal Modes

The key to understanding the dynamics of this time-invariant linear system is by means of its normal modes

$$q(t) = f(t)\ x$$

Eachone satisfies

$$\underbrace{\frac{\ddot{f}}{f}}\,Bx = -Ax$$

independent of time $\Longrightarrow \frac{\ddot{f}}{f} = const \equiv -\lambda$

Thus one has

a) $\ddot{f} + \lambda f = 0$

b) $Ax = \lambda Bx$

c.) Eigenvalue Problem

(1) The eigen value equation

$$(A-\lambda B)x = 0$$

has non-trivial solutions only for those values of λ which are the roots

$$\lambda = \lambda_1, \ldots, \lambda_n$$

of the secular polynomial

$$\det|A-\lambda B| = 0$$

(2) The corresponding eigenvectors are the elements of the nullspaces

$$\mathcal{N}(A-\lambda_i B) \qquad i = 1,2,\ldots,n$$

They satisfy

$$A\vec{x}_i = \lambda_i B\vec{x}_i \qquad i = 1,2,\ldots,n$$

Question: Are the vectors $\vec{x}_i$ and $\vec{x}_j$ orthogonal,

$$\langle \vec{x}_i, \vec{x}_j \rangle = 0$$

when $\lambda_i \neq \lambda_j$?

Answer: NO; in general $x_i^H x_j \neq 0$

However, let us introduce the

Definition:

$$\langle \vec{x}, \vec{y} \rangle_B = x^H B y,$$

is an inner product, the B-inner product, whenever B is positive definite, i.e. $B^H = B$ and $x^H B x > 0 \quad \forall x \neq 0$

(3) B – Orthonormality

Consider two different eigenvalues

$\lambda_i \neq \lambda_j$ of the eigenvalue equation

$$Ax = \lambda B x$$

One has the following

Theorem

Let $A x_i = \lambda_i B x_i$ and $A x_j = \lambda_j B x_j$

Then $\quad x_j^H B x_i = 0$ whenever $\lambda_i \neq \lambda_j$.

Proof:

Following Properties 1 and 2 on page 35.6

one has

1. $x^H A x$ is real
   $x^H B x$ is real and positive

2. $Ax = \lambda Bx$ has real eigenvalues.
   Indeed, using item 1. above, one finds. $x^T A x = \lambda x^T B x \Rightarrow \lambda$ is real

3. $x_j^H A x_i = \lambda_i x_j^H B x_i ; \quad x_i^H A x_j = \lambda_j x_i^H B x_j \quad (*)$

$(A^H x_j)^H x_i = \lambda_i x_j^H B x_i$

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$x_i^H A x_j = \overline{\lambda_i x_j^H B x_i}$

$= \bar{\lambda}_i \overline{(B^H x_j)^H x_i}$

$= \bar{\lambda}_i x_i^H B^H x_j$

$x_i^H A x_j = \lambda_i x_i^H B x_j \quad (**)$

Subtract Eq. (**) from Eq. (*) above and obtain

$$0 = \underbrace{(\lambda_j - \lambda_i)}_{\neq 0} x_i^H B x_j$$

$\therefore$

$$x_i^H B x_j = \langle x_i, x_j \rangle_B$$

i.e. x_i and x_j are B-orthogonal.

Using B-normalization, one obtains

$$\boxed{x_i^H B x_j = \delta_{ij}}$$

i.e. $\{x_i\}$ form a B-orthonormal eigenbasis

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E. Geometrization via Concentric Ellipses.

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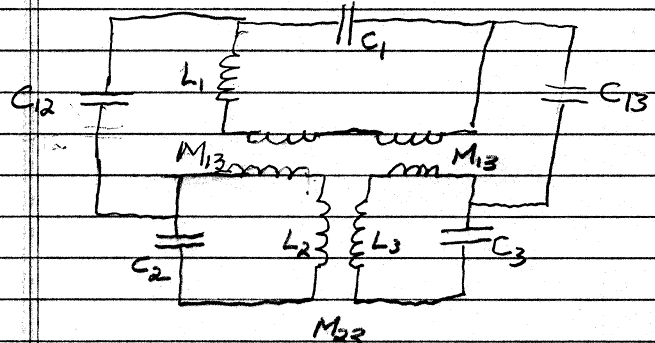

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One has the following

Theorem

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Then $\quad x_j^H B x_i = 0$ whenever $\lambda_i \neq \lambda_j$.

Proof:

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one has

Step 1. $x^H A x$ is real

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Step 3. $\quad x_j^H A x_i = \lambda_i x_j^H B x_i; \quad x_i^H A x_j = \lambda_j x_i^H B x_j \quad (*)$

$$(A^H x_j)^H x_i = \lambda_i x_j^H B x_i$$

$$(Ax_j)^H x_i = \quad "$$

$$x_i^H A x_j = \overline{\lambda_i x_j^H B x_i}$$

$$= \bar{\lambda}_i \overline{(B^H x_j)^H x_i}$$

$$= \bar{\lambda}_i \, x_i^H B^H x_j$$

$$x_i^H A x_j = \lambda_i \, x_i^H B x_j \quad (**)$$

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D.) Solution to the Eigenvalue Problem: Its Matrix Formulation.

The B-orthonormal eigenvectors x_i of the eigenvalue problem

$$Ax = \lambda Bx \qquad (*)$$

lead to the diagonalizing matrix

$$S = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}. \qquad (**)$$

Its columns are B-orthonormal. Consequently it satisfies

$$\boxed{S^T B S = I} \qquad (***)$$

It also diagononalizes A. Indeed, it satisfies

$$AS = BS \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \equiv BS\Lambda$$

which is Eq. (*) written in terms of the eigenvectors in Eq. (**). Multiplying

on the left by S^T one find with the help of Eq. (***) on page 41.7 that

$$\boxed{S^T A S = \Lambda} \qquad (\star)$$

E.) <u>Geometrization via Concentric Ellipses.</u>

The diagonalizing matrix S also geometrizes $Ax = \lambda Bx$. It does this by furnishing the transformation which relates the given coordinates $\{x^i\}$ to the new ones $\{y^i\}$,

$$x \equiv \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix} = S \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix} \equiv Sy$$

This coordinate transformation diagonalizes both quadratic forms

$$x^T A x \quad \underline{\text{and}} \quad x^T B x$$

simultaneously. Indeed one has

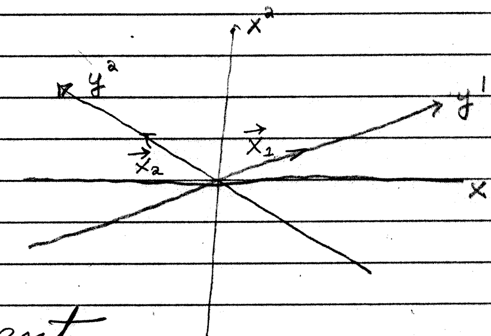
$$x^T A x = y^T S^T A S y = y^T \Lambda y$$
$$= \lambda_1 (y^1)^2 + \lambda_2 (y^2)^2 + \cdots + \lambda_n (y^n)^2,$$

$$x^T B x = y^T S^T B S y = (y^1)^2 + (y^2)^2 + \cdots + (y^n)^2$$

The columns of

$$S = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}$$

are the vectors which point along the new coordinate axes



<u>Comment</u>

Note that the y^1-axis and the y^2-axis are not orthogonal in the standard sense. This is because w.r.t. the standard inner product

$$x_i^T x_j \neq \delta_{ij}.$$

In fact, relative to the old coordinate system $\{x^i\}$ the new coordinate system

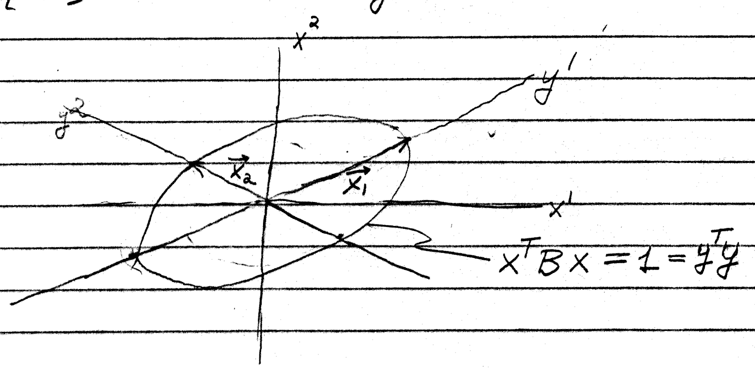
$\{y^i\}$ is one which is oblique. However, the y^1-axis and the y^2-axis <u>are</u> orthogonal with respect to the B-inner product. This is because

$$x_i^T B x_j = \delta_{ij}$$

The $x^T B x = 1$ isogram of

$$x^T B x = \sum \sum x^i B_{ij} x^j$$

is an ellipse relative to the old $\{x^i\}$ coordinate system.



However, it is a circle with respect the B-inner product because

$$x^T B x = y^T y.$$

The isograms

$$x^T A x = \lambda_1$$
$$\vdots$$
$$x^T A x = \lambda_n$$

of

$$x^T A x = \sum \sum x^i A_{ij} x^j$$

are also ellipses (relative the $\{x^i\}$ coordinate system) when A is positive definite.