LECTURE 41

I. Simultaneous Diagonalization of Two Quadratic Forms

A. General Coupled System

B. Normal Modes

C. Eigenvalue Problem

(1) Eigenvalue equation
(2) Eigenvectors
(3) B-orthonormality

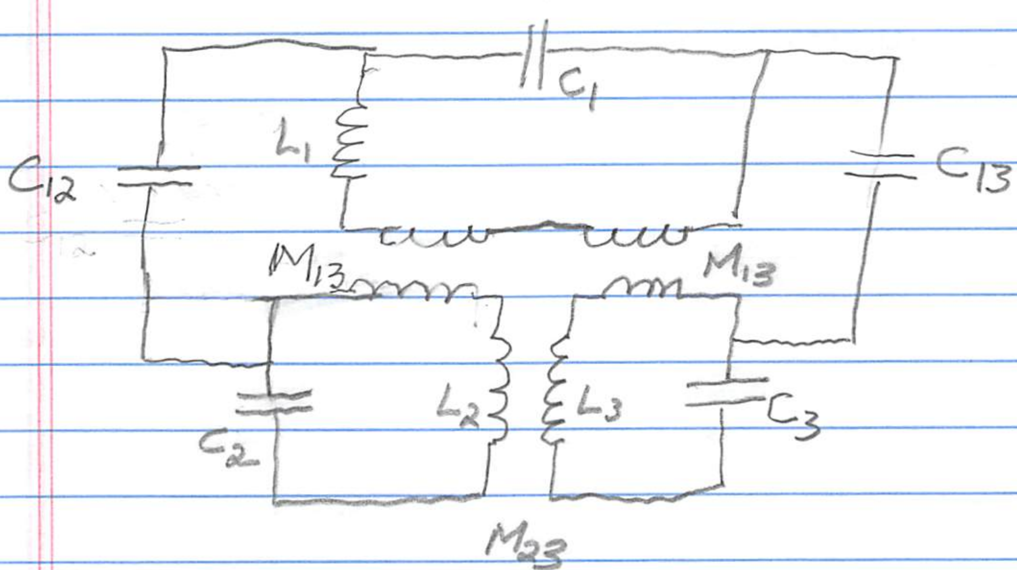
D. Solution to the Eigenvalue Problem: Its Matrix Formulation.

E. Geometrization via Concentric Ellipses.

I. SIMULTANEOUS DIAGONALIZATION OF TWO QUADRATIC FORMS.

A) General Coupled System:

Consider the equation of motion for the following system of inductively coupled networks



The equations of motion for this system are

$$\begin{bmatrix} L_1 & M_{12} & M_{13} \\ M_{12} & L_2 & M_{23} \\ M_{13} & M_{23} & L_3 \end{bmatrix} \frac{d^2}{dt^2} \overbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}}^{\ddot{q}(t)} + \begin{bmatrix} \frac{1}{C_1} & \frac{1}{C_{12}} & \frac{1}{C_{13}} \\ \frac{1}{C_{12}} & \frac{1}{C_2} & 0 \\ \frac{1}{C_{13}} & 0 & \frac{1}{C_3} \end{bmatrix} \overbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}}^{q(t)} = 0$$

Here $\{L_i\}$ are the self-inductances

$\{M_{ij} = M_{ji}\}$ are the mutual inductances

$\{C_i\}$ are capacitances

$\{C_{ij}=C_{ji}\}$ are mutual capacitances.

$\{I_i\}$ are the currents.

I.) More generally, we consider the following generalization of Eq. (*) on P 39,10

$$B\ddot{q} + Aq = 0.$$

Here $A^T = A$,
$B^T = B$,
$q^T B q > 0 \quad \forall$ vectors q.
that is B is a positive definite matrix.

B.) Normal Modes

C) The key to understanding the dynamics of this time-invariant linear system is by means of its normal modes

$$q(t) = f(t)\, x$$

Eachone satisfies

$$\underbrace{\frac{\ddot{f}}{f}}\, B x = -A x$$

independent of time $\Longrightarrow \frac{\ddot{f}}{f} = \text{const} \equiv -\lambda$

Thus one has

a) $\ddot{f} + \lambda f = 0$

b) $A x = \lambda B x$

c) Eigenvalue Problem

(1) The eigen value equation

$$(A - \lambda B) x = 0$$

has non-trivial solutions only for those values of λ which are the roots

$$\lambda = \lambda_1, \ldots, \lambda_n$$

of the secular polynomial

$$\det |A - \lambda B| = 0$$

(2) The corresponding eigenvectors are the elements of the nullspaces

$$\mathcal{N}(A-\lambda_i B) \qquad i=1,2,\ldots,n$$

They satisfy

$$A\vec{x}_i = \lambda_i B\vec{x}_i \qquad i=1,2,\ldots,n$$

Question: Are the vectors $\vec{x}_i$ and $\vec{x}_j$ orthogonal,

$$\langle \vec{x}_i, \vec{x}_j \rangle = 0$$

when $\lambda_i \neq \lambda_j$?

Answer: NO; in general $x_i^H x_j \neq 0$.

However, let us introduce the

Definition:

$$\langle \vec{x}, \vec{y} \rangle_B = x^H B y,$$

is an inner product, the B-inner product, whenever B is positive definite, i.e. $B^H = B$ and $x^H B x > 0 \quad \forall\, x \neq 0$

(3) <u>B-Orthonormality</u>

Consider two different eigenvalues

$\lambda_i \neq \lambda_j$ of the eigenvalue equation

$$Ax = \lambda Bx$$

One has the following

<u>Theorem</u>

Let $Ax_i = \lambda_i B x_i$ and $Ax_j = \lambda_j B x_j$ and $\lambda_i \neq \lambda_j$

Then $\boxed{x_j^H B x_i = 0}$ whenever $\lambda_i \neq \lambda_j$.

Use same line of reasoning as on P35.7

<u>Proof</u>:

Following Properties 1 and 2 on page 35.6

one has

Step 1. $x^H A x$ is <u>real</u>

$x^H B x$ is <u>real</u> and <u>positive</u>

Step 2. $Ax = \lambda Bx$ has real eigenvalues.

Indeed, using item 1. above, one finds. $x^T A x = \lambda x^T B x \Rightarrow$ <u>λ is real</u>

Reminder: $\overline{\langle x, y\rangle} = \langle y, x\rangle$

$\overline{x^H y} = y^H x$

Step 3. $(x_j^H A) x_i = \lambda_i x_j^H B x_i$; $x_i^H A x_j = \lambda_j x_i^H B x_j$ (*)

$$(A^H x_j)^H x_i = \lambda_i (B^H x_j)^H x_i$$

$A^H = A$

$$(A x_j)^H x_i = \lambda_i (B^H x_j)^H x_i$$

Take complex conjugate:

$$x_i^H A x_j = \lambda_i (B^H x_j) x_i$$

$$= \bar{\lambda}_i \, x_i^H B^H x_j$$

$$x_i^H A x_j = \lambda_i \, x_i^H B x_j \quad (**)$$

Subtract Eq. (**) from Eq. (*) above and obtain

$$0 = \underbrace{(\lambda_j - \lambda_i)}_{\neq 0} x_i^H B x_j$$

$\therefore$

$$x_i^H B x_j \equiv \langle x_i, x_j\rangle_B = 0 \quad \text{whenever } i \neq j$$

i.e. x_i and x_j are B-orthogonal.
Using B-normalization, one obtains

$$\boxed{x_i^H B x_j = \delta_{ij}}$$

i.e $\{x_i\}$ form a B-orthonormal eigenbasis

D.) Solution to the Eigenvalue Problem: Its Matrix Formulation.

The B-orthonormal eigenvectors x_i of the eigenvalue problem

$$A x = \lambda B x \qquad (*)$$

lead to the diagonalizing matrix

$$S = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}. \qquad (**)$$

Its columns are B-orthonormal. Consequently it satisfies

$x_i B x_j = \delta_{ij}$ $$\boxed{S^T B S = I} \quad (\text{"B-orthonormality"}) \quad (***)$$

It also diagononalizes A. Indeed, it satisfies

$$AS = BS \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \equiv BS\Lambda$$

which is Eq. (*) written in terms of the eigenvectors in Eq. (**). Multiplying on the left by S^T one find with the help of Eq. (***) on page 41.7 that

$$\boxed{S^T A S = \Lambda} \qquad (\star)$$

Thus S diagonalizes both A and B

E.) Geometrization via Concentric Ellipses.

(i) Simultaneous Diagonalization of 2 Quadratic Forms.

The diagonalizing matrix S also geometrizes $Ax = \lambda Bx$. It does this by furnishing the transformation which relates the given coordinates $\{x^i\}$ to the new coordinates $\{y^i\}$

$$x \equiv \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix} = S \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix} \equiv Sy$$

This coordinate transformation diagonalizes both quadratic forms

$$x^T A x \quad \text{and} \quad x^T B x$$

simultaneously. Indeed one has

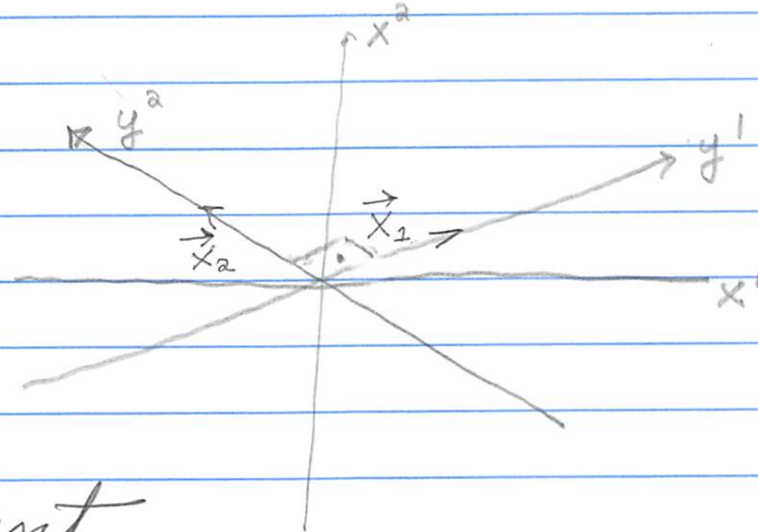
$$x^T A x = y^T S^T A S y = y^T \Lambda y$$
$$= \lambda_1 (y^1)^2 + \lambda_2 (y^2)^2 + \cdots + \lambda_n (y^n)^2,$$

$$x^T B x = y^T S^T B S y = (y^1)^2 + (y^2)^2 + \cdots + (y^n)^2$$

The columns of

$$S = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}$$

are the vectors which point along the new coordinate axes



Comment

Note that the y^1-axis and the y^2-axis are not orthogonal in the standard sense.

This is because w.r.t. the standard inner product

$$x_i^T x_j \neq \delta_{ij}.$$

In fact, relative to the old coordinate system $\{x^i\}$ the new coordinate systems

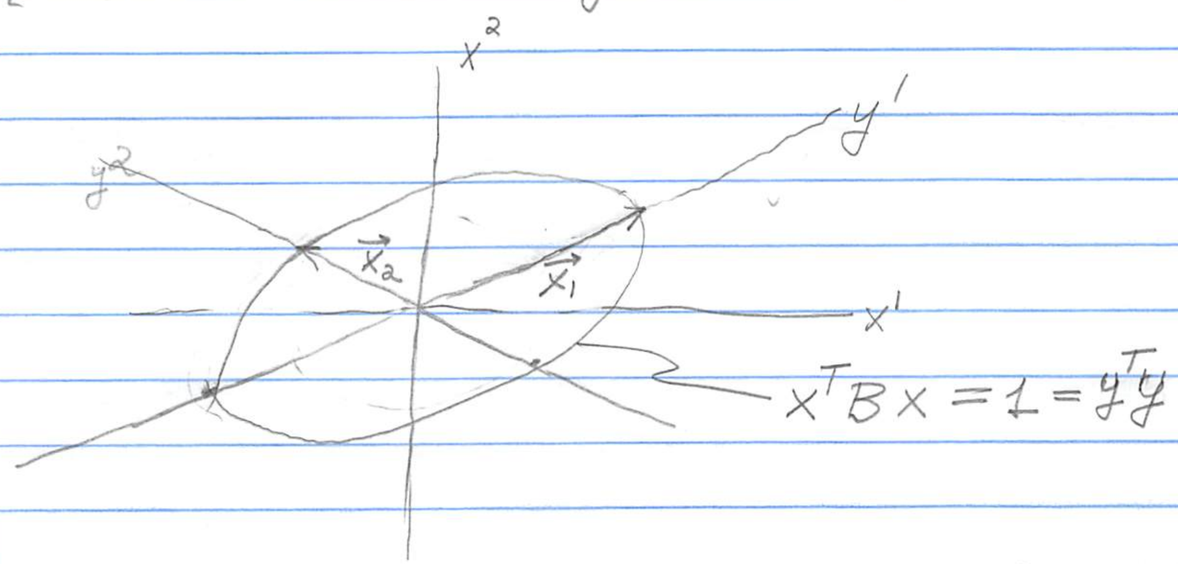
$\{y^i\}$ is one which is oblique. However, the y^1-axis and the y^2-axis are ortho-gonal with respect to the B-inner product. This is because

$$x_i^T B x_j = \delta_{ij}$$

The $x^T B x = 1$ isogram of

$$x^T B x = \sum \sum x^i B_{ij} x^j$$

is an ellipse relative to the old $\{x^i\}$ coordinate system.



Given the eigenvector induced change of coordinates

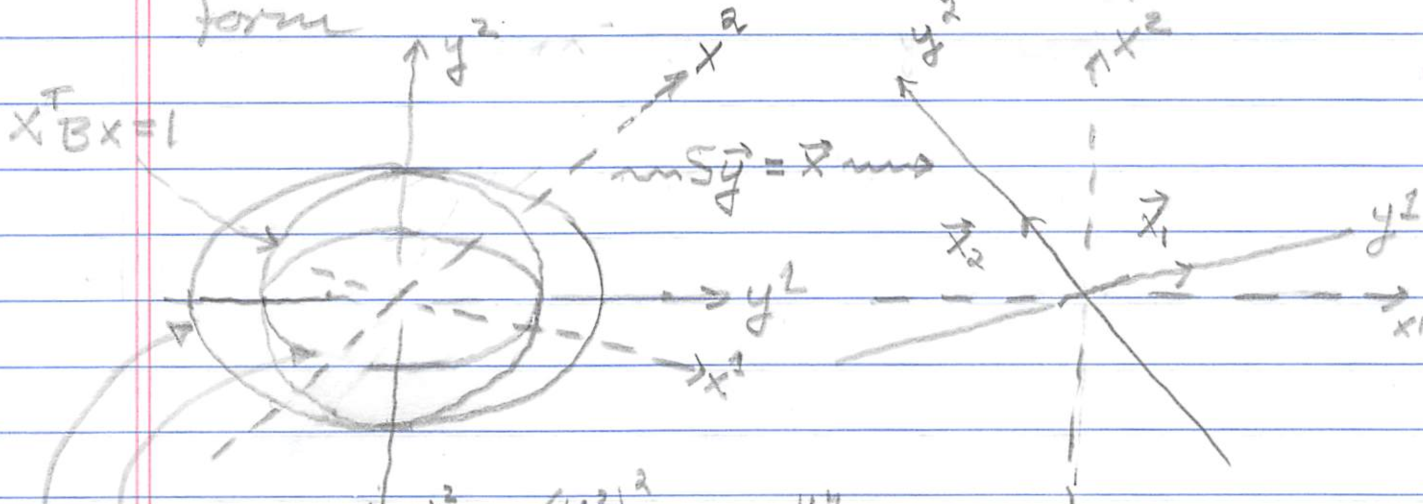
$$x=\begin{bmatrix}x^1\\x^2\\ \vdots\\x^n\end{bmatrix}=Sy=\begin{bmatrix}| & | & & |\\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n\\ | & | & & |\end{bmatrix}\begin{bmatrix}y^1\\y^2\\ \vdots\\y^n\end{bmatrix}$$

one obtains

$$x^TBx = y^T\underbrace{S^TBS}_{I}y = y^Ty = (y^1)^2+(y^2)^2+\cdots+(y^n)^2$$

$$x^TAx = y^T\underbrace{S^TAS}_{\Lambda}y = \lambda_1(y^1)^2+\lambda_2(y^2)^2+\cdots+\lambda_n(y^n)^2$$

Assuming $0<\lambda_1<\lambda_2\cdots<\lambda_n$,
consider the isograms of these two quadratic form



$$x^TAx=\frac{(y^1)^2}{\left(\frac{1}{\sqrt{\lambda_1}}\right)^2}+\frac{(y^2)^2}{\left(\frac{1}{\sqrt{\lambda_2}}\right)^2}+\cdots\frac{y^n}{\left(\frac{1}{\sqrt{\lambda_n}}\right)^2}=\lambda_1$$

$$=\left(\frac{y^1}{1}\right)^2+\left(\frac{y^2}{\sqrt{\lambda_1/\lambda_2}}\right)^2+\cdots=1$$

$$x^TAx= \qquad =\lambda_2$$

$$x^TAx=\left(\frac{y^1}{\sqrt{\lambda_2/\lambda_1}}\right)^2+\left(\frac{y^2}{1}\right)^2+\cdots=1$$

Conclusion:

All eigenvalue ellipses are tangent to the unit circle.