# APPENDIX A To Lecture 41

# PERIODIC RESONANT SYSTEMS

Stage I: Setting up the equations

Stage II: Solving the equations using the cyclic symmetry of the resonant system

A. The cyclic permutation matrix

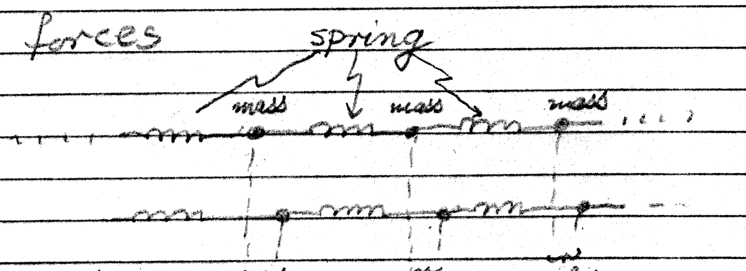
B. The normal modes

1. Time dependence

2. Amplitude profile

C. Toroidal Geometry

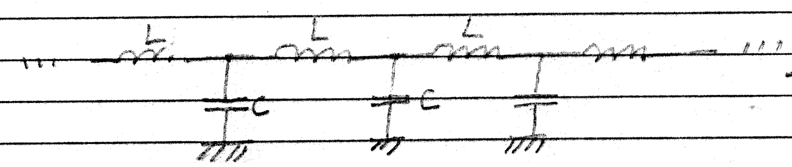
The strength of mathematics, including linear mathematics, lies in the fact that it has its roots in the physical world. It is difficult to point to a process more ubiquitous in science than one governed by a time-invariant linear system with a periodic structure. They range from the vibrations of a crystal consisting of an array of masses coupled by interatomic forces



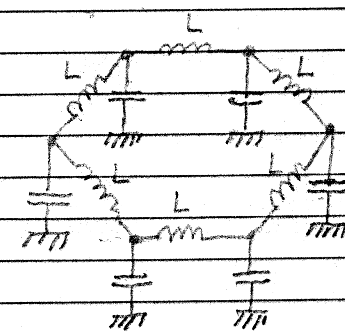

displacements: ..., $q^{k-1}$, $q^k$, $q^{k+1}$, ...,



an unlimited number of inductors L and capacitor C making up a transmission line




or a finite number of them making up a rotational structure such as



One arrives at the principles governing the nature and the behavior of these systems, and others like them, by means of an inductive process. In this particular circumstance it consists of a two-stage mental process:

I. Identify in quantitative form the causal relations that exist among the essential features of the system, i.e. set up the equations governing the system properties, and

II. Use the appropriate mathematical methods to solve these equations in order to recover, in actual numerical



form, the system properties that were originally used to form the concepts (charge, current, voltage, capacitance, inductance, charge con-sevation, Faraday's law of induction, etc), concepts and ideas that make up the essential features used in stage I.

The stage I mental process is one of induction (inferring generalization from particular instances). The stage II process is primarily one of deduction (applying a generalization to a particular case).

It is obvious that the results of the stage I mental process are fundamental to the possibility of the stage II process. Furthermore, given the amount of knowledge and the number of concepts (see the parenthetical list near the top of the previous page) that go into the inductive stage I process, it is not surpring that induction is considerably more challenging than deduction.



**STAGE I.** (Induction: Set up the Equations)

Consider a time-invariant linear system consisting of

(a) $N$ inductors $L_k$, $k=1,\ldots,N$,

with respective currents $j_k(t)$ at time $t$, and hence, by Faraday's Law of induction, having respective induced voltages

$$L_k \frac{d j_k(t)}{dt}$$

across each at time $t$, and

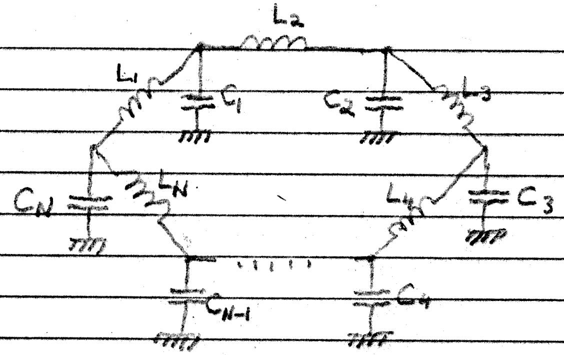
(b) $N$ capacitors $C_k$, $k=1,\ldots,N$,

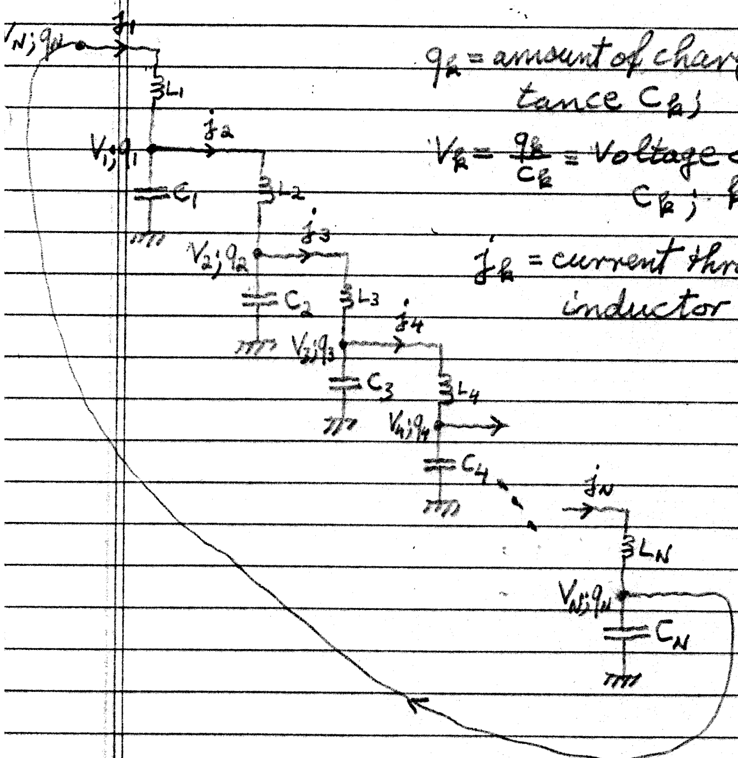
with respective charges $q_k(t)$ at time $t$, and hence having respective voltages

$$V_k(t) = \frac{q_k(t)}{C_k}$$

across each at time $t$.

The network diagram of inductors, capacitors and their connecting nodes are shown in two equivalent ways on the next page.




$q_k$ = amount of charge on capacitance $C_k$; $k = 1, \dots, N$.

$V_k = \dfrac{q_k}{C_k}$ = Voltage on capacitance $C_k$; $k = 1, \dots, N$.

$j_k$ = current through inductor $L_k$; $k = 1, \dots, N$.



The voltages at the nodes (heavy dots) are

$$V_N = L_1 \frac{dj_1}{dt} + \frac{q_1}{C_1} \quad \text{and} \quad V_N = \frac{q_N}{C_N}$$

$$V_1 = L_2 \frac{dj_2}{dt} + \frac{q_2}{C_2} \quad \text{and} \quad V_1 = \frac{q_1}{C_1}$$

$$V_2 = L_3 \frac{dj_3}{dt} + \frac{q_3}{C_3} \quad \text{and} \quad V_2 = \frac{q_2}{C_2}$$

$$V_3 = L_4 \frac{dj_4}{dt} + \frac{q_4}{C_4} \quad \text{and} \quad V_3 = \frac{q_3}{C_3}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$V_{N-1} = L_N \frac{dj_N}{dt} + \frac{q_N}{C_N} \quad \text{and} \quad V_{N-1} = \frac{q_{N-1}}{C_{N-1}}$$

Combine these two sets of equation to obtain

$$\frac{q_N}{C_N} = L_1 \frac{dj_1}{dt} + \frac{q_1}{C_1}$$

$$\frac{q_1}{C_1} = L_2 \frac{dj_2}{dt} + \frac{q_2}{C_2}$$

$$\frac{q_2}{C_2} = L_3 \frac{dj_3}{dt} + \frac{q_3}{C_3}$$

$$\frac{q_3}{C_3} = L_4 \frac{dj_4}{dt} + \frac{q_4}{C_4}$$

$$\vdots$$

$$\frac{q_{N-1}}{C_{N-1}} = L_N \frac{dj_N}{dt} + \frac{q_N}{C_N}$$

This is a linear system of N equations in 2N unknowns. To obtain N coupled equations in N unknowns

(i) take the time derivative, ~~and~~

(ii) use the principle of charge conservation at each node

$$\frac{dq_1}{dt} = j_1 - j_2$$

$$\frac{dq_2}{dt} = j_2 - j_3$$

$$\frac{dq_3}{dt} = j_3 - j_4$$

$$\vdots$$

$$\frac{dq_{N-1}}{dt} = j_{N-1} - j_N$$

$$\frac{dq_N}{dt} = j_N - j_1 \qquad (\leftarrow \text{Note!})$$

(iii) apply these equation to eliminate reference to the time derivatives of the q's. The resulting equations are



$$L_1 \frac{d^2 j_1}{dt^2} = \frac{1}{C_N} j_N - \left(\frac{1}{C_N} + \frac{1}{C_1}\right) j_1 + \frac{1}{C_1} j_2$$

$$L_2 \frac{d^2 j_2}{dt^2} = \frac{1}{C_1} j_1 - \left(\frac{1}{C_1} + \frac{1}{C_2}\right) j_2 + \frac{1}{C_2} j_3$$

$$L_3 \frac{d^2 j_3}{dt^2} = \frac{1}{C_2} j_2 - \left(\frac{1}{C_2} + \frac{1}{C_3}\right) j_3 + \frac{1}{C_3} j_4$$

$$L_4 \frac{d^2 j_4}{dt^2} = \frac{1}{C_3} j_3 - \left(\frac{1}{C_3} + \frac{1}{C_4}\right) j_4 + \frac{1}{C_4} j_5$$

$$\vdots$$

$$L_N \frac{d^2 j_N}{dt^2} = \frac{1}{C_{N-1}} j_{N-1} - \left(\frac{1}{C_{N-1}} + \frac{1}{C_N}\right) j_N + \frac{1}{C_N} j_1$$

This is coupled system of equations of the type.

$$B \frac{d^2 \vec{j}}{dt^2} = A \vec{j}$$

where $B$ is positive definite.

STAGE II (Deduction: Solve the system of differential eq'ns)

If $N \leq 4$, one can find the normal modes

$$\vec{J}_k(t) = e^{\lambda_k t}\, \vec{X}_k \qquad k = 1, \cdots, N \leq 4$$

by solving the eigenvalue problem

$$(A - \lambda B)\vec{X} = \vec{0}.$$

algebraically in terms of the roots of the characteristic polynomial

$$\det|A - \lambda B| = 0.$$

For $N > 4$ this algebraic approach is in general impossible because it involves a quintic polynomial.

However, when all the capacitances and all the inductances are equal,



$$C_1 = \cdots = C_N \equiv C; \quad L_1 = \cdots = L_N \equiv L$$

then one's focus has shifted to physical systems with an ubiquitous and directly observable property which is also physically and mathematically fundamental.

This system property, symmetry under cyclic permutations of its state vector

$$\vec{J} = \begin{bmatrix} J_1 \\ J_2 \\ \vdots \\ J_N \end{bmatrix} \rightsquigarrow T\vec{J} = \begin{bmatrix} J_2 \\ J_3 \\ \vdots \\ J_N \\ J_1 \end{bmatrix},$$

is defined by the statement that if $\vec{J}$ is a solution to

The matrix formulation of the system, Eq.(*) on page A41.10, with this property is

$$L\frac{d^2}{dt^2}\begin{bmatrix} j_1 \\ j_2 \\ \vdots \\ \vdots \\ j_{N-1} \\ j_N \end{bmatrix} = \frac{1}{C}\underbrace{\begin{bmatrix} -2 & 1 & 0 & \cdots & & 1 \\ 1 & -2 & 1 & & & 0 \\ 0 & 1 & -2 & & & 0 \\ & \vdots & & \ddots & & \vdots \\ & & & & -2 & 1 \\ 1 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix}}_{K}\begin{bmatrix} j_1 \\ j_2 \\ \vdots \\ \vdots \\ j_{N-1} \\ j_N \end{bmatrix}$$

or

$$L\frac{d^2\vec{j}(t)}{dt^2} = \frac{1}{C}K\vec{j} \qquad (1)$$

A) THE SYMMETRY PROPERTY

Consider the state vector $\vec{j}(t)$ subjected to the linear transformation

$$\vec{j} = \begin{bmatrix} j_1 \\ j_2 \\ \\ j_{N-1} \\ j_N \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ & & & 0 & 1 \\ 1 & \cdots & & 0 & 0 \end{bmatrix}\begin{bmatrix} j_1 \\ j_2 \\ \\ j_{N-1} \\ j_N \end{bmatrix} \equiv T\vec{j} = \begin{bmatrix} j_2 \\ j_3 \\ \\ j_N \\ j_1 \end{bmatrix}$$



The matrix T has several key properties:

(i) It is an orthogonal cyclic permutation matrix,

(ii) It takes solutions to Eq.(1) on page A 41.13 into solutions of the same system.

(Go to page A 41.15)

In other words if $\vec{j}$ is a solution to Eq.(1) on page A41.13, then so is $T\vec{j}$

$$L\frac{d^2(T\vec{j})}{dt^2} = \frac{1}{c}K(T\vec{j}) .$$

Thus $T$ expresses a symmetry property of the system.

This is because

$$TK = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & 0 \; 1 \\ 1 & 0 & 0 & & 0 \; 0 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 0 & \cdots & \\ & 1 & & -2 & 1 \\ 1 & \cdots & & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & \cdots & & 0 \\ 0 & 1 & & & 0 \\ \vdots & & & & \vdots \\ 1 & 0 & & & -2 \\ -2 & 1 & & & 1 \end{bmatrix}$$

$$KT = \begin{bmatrix} -2 & 1 & 0 & & 1 \\ 1 & -2 & 0 & \cdots & 0 \\ & & & & \\ 0 & & & -2 & 1 \\ 1 & & & 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & \\ & & & & \\ & & & & 0 \; 1 \\ 1 & 0 & 0 & & 0 \; 0 \end{bmatrix} = \text{same as above}$$

i.e. because

$$TK = KT.$$

(iii) The eigenvalues of $T$ are distinct. Indeed, consider the



eigenvalue problem

$$Tx = \eta x$$

hence

$$T^N x = \eta^N x$$

In light of the fact that $T^N = I$, one has $\eta^N = 1$

Consequently, the eigenvalues of $T$ are

$$\eta_k = e^{2\pi i k/N} \qquad k = 0, 1, \ldots, N-1$$

The corresponding eigenvectors of $T$ are

$$\vec{X}_k = \begin{bmatrix} e^{2\pi i \, 1 \cdot k/N} \\ e^{2\pi i \, 2 \, k/N} \\ e^{2\pi i \, 3 \, k/N} \\ \vdots \\ e^{2\pi i \, N \, k/N} \end{bmatrix} \qquad k = 0, 1, \ldots, N-1 \qquad (\star)$$

(iv) $TK = KT$ implies that each $\vec{X}_k$ is also an eigenvector of $K$:

$$K\vec{X}_k = \lambda_k \vec{X}_k \qquad \text{for some number } \lambda_k.$$

B) NORMAL MODES OF THE SYSTEM.

A normal mode of the time-invariant system has the form

$$\vec{J}(t) = f(t)\,\vec{x}\,; \quad (\vec{x} \text{ is indep. of } t);$$

all vector components have the same time dependence $f(t)$. A solution to Eq.(1) on A41.13 satisfies

$$L\frac{d^2 f}{dt^2}\vec{x} = \frac{f(t)}{C}K\vec{x} \longrightarrow LC\frac{d^2 f}{dt^2}\frac{1}{f}\vec{x} = K\vec{x}$$

The same reasoning as on page 39.11 lead to

$$\boxed{\frac{d^2 f}{dt^2} - \frac{\lambda}{LC} f = 0} \qquad (*)$$

where

$$\boxed{K\vec{x} = \lambda\vec{x}} \qquad (**)$$

1) The time dependence $f(t)$ of a normal is



$$f(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$$

where $\omega = \sqrt{\frac{-\lambda}{LC}}\,;\quad \omega^2 = \frac{-\lambda}{LC}$ (1)

2) This normal mode frequency as well as the amplitude profile $\vec{x}$ are determined by the boxed eigenvalue Eq.(**) on page A41.17, i.e. by

$$K\vec{x} = \lambda\vec{x}.$$

The eigenvectors $\vec{x}_k$ ($k = 0, 1, \ldots, N-1$) of $K$ are given by Eq.(*) on page A41.16.

Applying $K$ each of them yields

$$K\vec{x}_k = \begin{bmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & \vdots \\ \vdots & & & -2 & 1 \\ 1 & \cdots & & 1 & -2 \end{bmatrix} \begin{bmatrix} e^{2\pi i 1 k/N} \\ e^{2\pi i 2 k/N} \\ e^{2\pi i 3 k/N} \\ \vdots \\ e^{2\pi i (N-1) k/N} \\ e^{2\pi i N k/N} \end{bmatrix}$$

yields

$$K\vec{x}_k = \underbrace{\left(-2+e^{2\pi i k/N}+e^{-2\pi i k/N}\right)}_{-2+2\cos(2\pi k/N)} \begin{bmatrix} e^{2\pi i k/N} \\ e^{2\pi i 2k/N} \\ e^{2\pi i 3k/N} \\ \vdots \\ e^{2\pi i (N-1)k/N} \\ e^{2\pi i N k/N} \end{bmatrix}$$

$$= \left(-4\sin^2\frac{\pi k}{N}\right)\vec{X}_k \qquad k = 0,1,\dots,N-1$$

In light of the eigenvalue equation at the bottom of page A41.16, the eigenvalues of $K$ are

$$\lambda_k = -4\sin^2\frac{\pi k}{N}$$

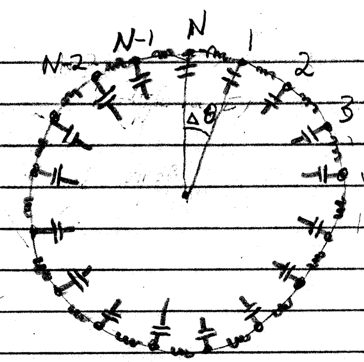
and the resonance frequencies, Eq.(1) on page A41.18, of the normal modes,

$$\vec{j}_k(t) = \left(c_1 e^{i\omega_k t} + c_2 e^{-i\omega_k t}\right) \begin{bmatrix} e^{2\pi i k/N} \\ e^{2\pi i 2k/N} \\ \vdots \\ e^{2\pi i Nk/N} \end{bmatrix};$$

are

$$\omega_k = \frac{2}{\sqrt{LC}}\sin\frac{\pi k}{N} \qquad k = 0,1,\dots,N-1$$






Suppose one arranges the LC elements in the figure on page A41.7 on a torus separated by $\Delta\theta$ so that

$$\Delta\theta N = 2\pi$$

Letting $\Delta\theta k \equiv \theta_k$

one obtains for the normal mode profile

$$\vec{X}_k = \begin{bmatrix} e^{i\theta_k} \\ e^{2i\theta_k} \\ e^{3i\theta_k} \\ \vdots \\ e^{Ni\theta_k} \end{bmatrix} \qquad \theta_k = 0, \Delta\theta, \dots, (N-1)\Delta\theta$$

($= k^{th}$ inverse angular wave length)

A41.21

and for the normal mode frequency

$$\omega_k = \frac{2}{\sqrt{LC}} \sin \frac{\theta_k}{2}$$