LECTURE 4

Coordinte Representative of a vector relative to a given basis
(Existence & uniqueness)

Coordinates

Without numbers (i.e. scalars) the theory of vector spaces would be a mere aggregate of floating abstractions disconnected and detached from the world

Numbers are the means of quantifying and processing the evidence of the senses.

A given vector is subjected to a measuring process, i.e. by relating it a chosen standard, a chosen basis. The result of the measurement is expressed in terms of numbers relative to the basis vectors, which comprise the basis. The n measurements applied to each one of an aggregate of vectors get conceptualized (by omitting reference to any particular vector) into their coordinate representation

4.1

Example 1

What is a basis for each of the three vector spaces

a) $V_1 = \left\{ \begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} \right\}$; b) $V_2 = \{ a + b - bx + ax^3 \}$; c) $V_3 = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \right\}$

where a and b are scalars.

a) $$\begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{v_1} + b \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}}_{v_2} \equiv a v_1 + b v_2$$

$B_1 = \{ v_1, v_2 \}$ is a lin. indep. spanning set for V_1

b) $a + b - bx + ax^3 = a(1 + x^3) + b(1 - x)$

$B_2 = \{ 1 + x^3, 1 - x \}$ is a lin. indep. spanning set for V_2.

c) $$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$B_3 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ is a lin. indep. spanning set for V_3

I The Coordinate Representative of a Vector Relative to a Given Basis

Given a vector $p = a_0 + a_1 x + a_2 x^2$,

the application of the linear independence

! of Q on page ~~3.3~~ 3.6 to Eq.(*)

known

$$(1+x)r + (x+x^2)s + (1+x^2)t = a_0 + a_1 x + a_2 x^2$$

[By contrast, $(1+x)r + (x^2+x^2)s + (1-x^2)t = a_0 + a_1 x + a_2 x^2$ has no sol'n unless $a_2 = a_1 - a_0$]

lead to a very important result, namely

that the solution on p ~~3.3~~ 3.6

$$\boxed{\begin{aligned} r &= \tfrac{1}{2}(a_0 + a_1 - a_2) \\ s &= \tfrac{1}{2}(-a_0 + a_1 + a_2) \\ t &= \tfrac{1}{2}(a_0 - a_1 + a_2) \end{aligned}}$$

is unique.

The generalization of this result to an arbitrary vector space is expressed by the following "Representation Thm"

Theorem 4 (Vector uniquely represented in terms of a basis) ("The Representation Theorem")

Let $B = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be a basis for the vector space V.

Let $w \in V$ be any given vector in V.

Conclusion:

w has a unique representation relative to B, i.e.

there exist unique scalars $a_1, \ldots, a_p$ such

exist!

that $\vec{w} = a_1\vec{v}_1 + \cdots + a_p\vec{v}_p$

unique!

Comment 2: This theorem makes two strong claims, namely existence and uniqueness.

Comment 1: This theorem can be restated as follows:

For every vector w in V, there is exactly one way to write w as a lin. comb'n of the basis vectors in B.

"existence" "uniqueness"

Comment 1:

This theorem can be restated as follows:

For every vector w in W, $\underbrace{\text{there is}}_{\text{"exists"}}$ $\overbrace{\text{exactly one}}^{\text{"unique"}}$ way to write w as a linear combination of the basis vectors in B.

Comment 2:

This theorem makes <u>two</u> claims, namely <u>existence</u> and <u>uniqueness</u>.

Proof of existence:

B is a spanning set for $V \Longrightarrow$ for $w \in V$ $\exists$

scalars $a_1, \ldots, a_p$ such that

$$\boxed{w = a_1 v_1 + \cdots + a_p v_p} \qquad (1)$$

Proof of uniqueness:

Let $b_1, \ldots, b_p$ be another set of scalars with the property that

$$w = b_1 v_1 + \cdots + b_p v_p$$

Subtracting one finds

$$w - w = 0 = (a_1 - b_1) v_1 + \cdots + (a_p - b_p) v_p$$

B is a basis $\Rightarrow$ B is a linearly independent set

$\Longrightarrow$ $\quad a_1 - b_1 = 0, \ldots, a_p - b_p = 0$

i.e $\boxed{a_i = b_i}$ for $i = 1, \ldots, p$.

Thus Eq.(1) at the top of 3.11, namely,

$$w = \sum_{i=1}^{p} a_i v_i$$

is a unique representation of w indeed.

The importance of Theorem 4 is that any vector $w \in V$ determines and is determined by p scalars

$$a_1, \ldots, a_p$$

once a basis $B = \{v_1, \ldots, v_p\}$ is given (or has been chosen). INSERT on P4,6b This set of a scalars is a new concept, namely, the coordinate representative with respect to a basis. Explicitly, one has the following

NOTE TO LECTURER: (Needs elaboration) These scalars comprising the connecting link between abstract linear algebra and the evidence of the senses.

Definition 5 (Coordinates of a vector)

The scalars $a_1, \ldots, a_p$ are called the coordinates of w relative to

$$B = \{v_1, \ldots, v_p\} \subset V$$

INSERT for p 4.6

There is a 1-1 between vector w in V and

p-tuples $(a_1, \ldots, a_p)$ in R^p.

$\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}_B$ vs w!

In mathematical notation one writes them as the column array

$$[\vec{w}]_B \equiv \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}_B ,$$

the coordinate representative of w relative to B. The entries of the column array are the coordinates of w relative to B.

One needs to emphasize that the column vector $\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}_B \in R^p$ is quite distinct from the vector $w \in V$ (although, as we shall find on page 4,13, they correspond to one another once a basis has been specified). This fact is illustrated by the following

Example 2

a) Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\} = M_{22}$, the vector space of 2×2 matrices.

b) Let $B = \left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

be the <u>standard</u> basis for V

c) Let $C = \left\{ F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, F_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, F_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, F_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

be another basis for V

d) Let $w = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a known element of V

<u>FIND</u> the coordinate representative of w (i) relative to B and (ii) relative to C.

Finding these representatives is 3 step process:

Step 1:

Expand w in terms of the basis elements.

(i) Relative to B one has by inspection

$$w = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = aE_{11} + bE_{12} + cE_{21} + dE_{22}$$
$$= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(ii) Relative to C one sets

$$r\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + s\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + t\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + u\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} (= w)$$

and solves for r, s, t, u and obtains

$$\begin{aligned} r + s + t + u &= a \\ s + t + u &= b \\ t + u &= c \\ u &= d \end{aligned}$$

or

$$\begin{aligned} u &= d \\ t &= c - d \\ s &= b - c \\ r &= a - b \end{aligned}$$

Step 2

Read out the two coordinate representatives

$$[w]_B = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_B \qquad \text{(relative to B)}$$

$$[w]_C = \begin{bmatrix} a-b+c \\ b-c \\ c-d \\ d \end{bmatrix}_C \qquad \text{(relative to C)}$$

Comment 1:

The key idea is that even though one has coordinate 4-tuples in both cases, the respective coordinates have entirely different meaning because they refer to entirely different bases.

4.11

From the perspective of the relationship of w to physical world, a chosen basis is the standard relative to which one measures w. The coordinate components of w are the result of this measurement. Thus, if one changes the standard of measurement, then the result of the measurement will necessarily also change.