LECTURE 4

Coordinate Representative of a vector relative to a given basis
(Existence & uniqueness)

Coordinates
Without numbers the theory of vector spaces would be a mere aggregate of floating abstractions disconnected and detached from the world. Numbers are the means of quantifying and processing the evidence of the senses.

A given vector is subjected to a measuring process, i.e. by relating it to a chosen standard, a chosen basis. The result of the measurement is expressed in terms of numbers relative to the basis vectors, which comprise the basis. Then measurements applied to each one of an aggregate...
of vectors get conceptualized (by omitting reference to any particular vector) into their coordinate representation.
Example:

What is a basis for each of the three vector spaces

\[ V_1 = \left\{ \begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} \right\}; \quad b) V_2 = \left\{ a + b - bx + ax^3 \right\}; \quad c) V_3 = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \right\} \]

where \( a \) and \( b \) are scalars.

\begin{align*}
\text{a)} \; \begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} &= a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = a v_1 + b v_2 \\
B_1 &= \left\{ v_1, v_2 \right\} \text{ is a lin. indep. spanning set for } V_1
\end{align*}

\begin{align*}
\text{b)} \; a + b - b x - a x^3 &= a(1 + x^3) + b(1 - x) \\
B_2 &= \left\{ 1 + x^3, 1 - x \right\} \text{ is a lin. indep. spanning set for } V_2
\end{align*}

\begin{align*}
\text{c)} \; \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} &= a \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \\
B_3 &= \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ is a lin. indep. spanning set for } V_3
\end{align*}
The Coordinate Representation of a Vector Relative to a Given Basis

Given a vector \( p = a_0 + a_1 x + a_2 x^2 \), the application of the linear independence of \( \Omega \) on page 3.3 to Eq. (4) known

\[
(1+x) \alpha + (x+x^2) \beta + (1+x^2) \gamma \Rightarrow \alpha = a_0 + a_1 x + a_2 x^2
\]

[By contrast, \((1+x)^2 + (x^2+x^3) \alpha + (1-x^2) \gamma = a_0 + a_1 x + a_2 x^2 \) has no solution unless \( a_2 = a_1 a_2 \]].

lead to a very important result, namely

that the solution on page 3.3

\[
\begin{align*}
\gamma &= \frac{1}{2} (a_0 + a_1 - a_2) \\
\beta &= \frac{1}{2} (-a_0 + a_1 + a_2) \\
\gamma &= \frac{1}{2} (a_0 - a_1 + a_2)
\end{align*}
\]

is unique.

The generalization of this result to an arbitrary vector space is...
expressed by the following "Representation Thm"

**Theorem 4** (Vector uniquely represented in terms of a basis) ("The Representation Theorem")

Let \( B = \{ \vec{v}_1, \ldots, \vec{v}_p \} \) be a basis for the vector space \( V \).

Let \( \vec{w} \in V \) be any given vector in \( V \).

**Conclusion:**

\( \vec{w} \) has a unique representation relative to \( B \), i.e., there exist unique scalars \( a_1, \ldots, a_p \) such that

\[
\vec{w} = a_1 \vec{v}_1 + \cdots + a_p \vec{v}_p
\]

**Comment 1:** This theorem makes two strong claims namely existence and uniqueness.

**Comment 2:** This theorem can be restated as follows:

For every vector \( \vec{w} \) in \( V \), there is exactly one way to write \( \vec{w} \) as a linear combination of the basis vectors in \( B \).
Comment 1:
This theorem can be restated as follows:

For every vector \( w \) in \( W \), there is exactly one way to write \( w \) as a linear combination of the basis vectors in \( B \).

Comment 2:
This theorem makes two claims, namely existence and uniqueness.
Proof of existence:

B is a spanning set for \( V \) \( \Rightarrow \) for \( w \in V \) \( \exists \) scalars \( a_1, \ldots, a_p \) such that

\[
\begin{align*}
w &= a_1v_1 + \cdots + a_pv_p
\end{align*}
\]

Proof of uniqueness:

Let \( b_1, \ldots, b_p \) be another set of scalars with the property that

\[
w = b_1v_1 + \cdots + b_pv_p
\]

Subtracting one finds

\[
w - w = 0 = (a_1 - b_1)v_1 + \cdots + (a_p - b_p)v_p
\]

\( B \) is a basis \( \Rightarrow \) \( B \) is a linearly independent set

\[
\Rightarrow \quad a_i - b_i = 0, \quad \ldots, \quad a_p - b_p = 0
\]

i.e. \( a_i = b_i \) for \( i = 1, \ldots, p \)

Thus Eq. (1) at the top of 3.11, namely

\[
w = \sum_{i=1}^{p} a_i v_i
\]

is a unique representation of \( w \) indeed.
The importance of Theorem 4 is that any vector $w \in V$ determines and is determined by $p$ scalars $a_1, \ldots, a_p$

Once a basis $B = \{v_1, \ldots, v_p\}$ is given (or has been chosen), this set of scalars is a new concept, namely, the coordinate representative with respect to a basis. Explicitly, one has the following.

**Definition 5 (Coordinates of a vector)**
The scalars $a_1, \ldots, a_p$ are called the coordinates of $w$ relative to $B = \{v_1, \ldots, v_p\} \subseteq V$. 
There is a 1-1 between vector \( w \) in \( V \) and \( p \)-tuples \( (a_1, \ldots, a_p) \) in \( \mathbb{R}^p \).
In mathematical notation one writes them as the column array

\[
\begin{bmatrix}
a_1 \\
\vdots \\
a_p
\end{bmatrix}_B
\]

the coordinate representative of \( \mathbf{w} \) relative to \( \mathbf{B} \). The entries of the column array are the coordinates of \( \mathbf{w} \) relative to \( \mathbf{B} \).

One needs to emphasize that the column vector \( \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} \in \mathbb{R}^p \) is quite distinct from the vector \( \mathbf{w} \in \mathbf{V} \) (although, as we shall find on page 4.13, they correspond to one another once a basis has been specified). This fact is illustrated by the following
Example 2

d) Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\} = M_{22}$, the vector space of $2 \times 2$ matrices.

e) Let $B = \left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ be the standard basis for $V$

f) Let $C = \left\{ F_{1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, F_{2} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, F_{3} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, F_{4} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ be another basis for $V$

g) Let $w = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an element of $V$

FIND the coordinate representative of $w$ (i) relative to $B$ and (ii) relative to $C$. Finding these representatives is a 3 step process!
Step 1: Expand \( w \) in terms of the basis elements.

(i') Relative to \( B \) one has by inspection

\[
w = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

(ii') Relative to \( C \) one sets

\[
r \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} r \\ s \\ t \end{bmatrix}
\]

and solves for \( r, s, t, u \) and obtains

\[
\begin{align*}
r + s + t + u &= a \\
s + t + u &= b \\
t + u &= c \\
u &= d
\end{align*}
\]

or

\[
\begin{align*}
u &= d \\
t &= c - d \\
s &= b - c + d \\
r &= a - b + c
\end{align*}
\]
Step 2
Read out the two coordinate representations

\[ [w]_B = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_B \quad \text{(relative to B)} \]

\[ [w]_C = \begin{bmatrix} a-b+c \\ b-c \\ c-d \\ d \end{bmatrix}_C \quad \text{(relative to C)} \]

Comment 1:
The key idea is that even though one has coordinate 4-tuples in both cases, the coordinates have entirely different meaning because they refer to entirely different bases.
From the perspective of the relationship of $w$ to physical world, a chosen basis is the standard relative to which one measured $w$. The coordinate components of $w$ are the result of this measurement. Thus, if one changes the standard of measurement, then the result of the measurement will necessarily also change.