LECTURE 5

1. Basis-induced correspondence between V and R^p as structure preserving
2. Preservation of linear independence and linear dependence of sets of vectors
3. Isomorphism and its basis indendence.

The implicit basis-induced one-to-one correspondence between V and R^p, which is expressed by Theorem 4 on page 4.3, is the doorway between

(i) the highly abstract (and hence widely applicable) geometric approach to grasping the nature of the world and

(ii) the computational approach (i.e. via the linear algebra of R^p) which concretizes the nature of the world in always the same way, namely by means of numbers as is done with simulation software (e.g. MATLAB) on a computer.

"Isomorphism" Between V and R^p

Given a basis $B = \{v_1, \ldots, v_p\} \subset V$ for vector space V one has

$$w = a_1 v_1 + \cdots + a_p v_p$$

with uniquely defined scalars $\{a_1, \ldots, a_p\}$.

Thus one has

$$w \rightsquigarrow \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} \equiv [w]_p$$

$$V \longrightarrow R^p$$

Conversely,

$$\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} \rightsquigarrow a_1 v_1 + \cdots + a_p v_p \equiv w$$

$$R^p \longrightarrow V$$

Thus one has the following

~~4.13~~
5.3

Proposition ($V \longleftrightarrow R^p$ is structure preserving)

a) A basis $B = \{v_1, \ldots, v_p\} \subset V$ for vector space V

induces the 1-1 mapping

$$V \longleftrightarrow R^p$$
$$w \leftrightsquigarrow [w]_B \equiv \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix},$$

namely for every w there is a unique $[w]_B$ and for every $[w]_B$ there is a unique w:

$$w = a_1 v_1 + \cdots + a_p v_p.$$

b) This mapping preserves structure,

namely
(i) addition
(ii) scalar multiplication
(iii) linear combinations
(iv) linear independence

i.e. one has

$$\left.\begin{matrix}(i)\\(ii)\\(iii)\end{matrix}\right\} \quad [c_1 w_1 + c_2 w_2]_B = c_1 [w_1]_B + c_2 [w_2]_B \qquad (*)$$

(iv)} $\{w_1, \ldots, w_k\}$ is linearly independent (dependent)

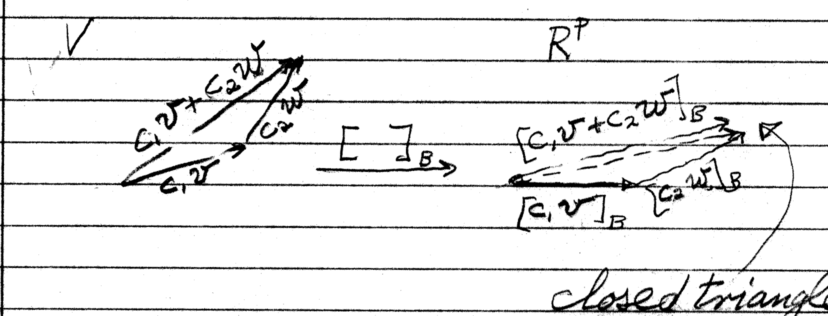
$\Leftrightarrow \{[w_1]_B, \ldots, [w_k]_B\}$ is lin. indep. (depend.)

5.4

Comment:

Q: What is the geometrical meaning of Eq.(*)?

A: The correspondence $[\ ]_B$ maps closed triangles in V to closed triangles in R^p:



Algebraically we have the following

1. Let $\{v_1, \ldots, v_p\}$ be a basis for V then

$$[c_1 v + c_2 w]_B = [c_1(a_1 v_1 + \cdots + a_p v_p) + c_2(b_1 v_1 + \cdots + b_p v_p)]_B$$
$$= [(c_1 a_1 + c_2 b_1) v_1 + \cdots + (c_1 a_p + c_2 b_p) v_p]_B$$
$$= \begin{bmatrix} c_1 a_1 + c_2 b_1 \\ \vdots \\ c_1 a_p + c_2 b_p \end{bmatrix}_B = c_1 \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}_B + c_2 \begin{bmatrix} b_1 \\ \vdots \\ b_p \end{bmatrix}_B = c_1 [v]_B + c_2 [w]_B$$

Q.E.D.

Preservation of linear structure by a basis-induced representation means that all vector space operations in V are mirrored in R^p. This means that all computations in V can be achieved by doing them in R^p.

These computation are done in terms of matrices.

Computations involving linear (in)dependence illustrate this principle

Theorem 5 (Preservation of linear (in)dependence)

Let $B = \{v_1, \dots, v_p\}$ be a basis for V

Then

$\{u_1, \dots, u_m\}$ is lin. indep in $V \Leftrightarrow \{[u_1]_B, \dots, [u_m]_B\}$ is lin. indep. in R^p.

Know why this is so.

Comment 1: This conclusion is equivalent to:

$\{u_1, \dots, u_m\}$ is lin. dep in $V \Leftrightarrow \{[u_1]_B, \dots, [u_m]_B\}$ is lin. dep. in R^p.

Comment 2: This theorem is validated by starting in V, but doing the computation in R^p.

proof: 1) Consider the following vectorial equation in V:

$$u_1 c_1 + u_2 c_2 + \cdots + u_m c_m = \vec{0} \qquad (\star)$$

2) In R^p we have

by Eq. $(\star)$ on P5.3

$$[u_1]_B c_1 + \cdots + [u_m]_B c_m = [\vec{0}]_B$$

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{p1} \end{bmatrix} c_1 + \cdots + \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{pm} \end{bmatrix} c_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & & \vdots \\ a_{p1} & \cdots & a_{pm} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (\star\star)$$

This is p equations in m unknowns, where $p = \#$ of basis vectors

Comment: If the columns of the matrix form an independent set then $c_1 = \cdots = c_m = 0$ is the only solution to $(\star)$. If the columns of the matrix form a dependent set then $\exists$ a non-trivial solution $[c_1, \dots, c_m]^t$ to Eq. $(\star)$,

Conclusion:

$\{u_1, \dots, u_m\}$ forms a l. indep. set $\Leftrightarrow$ the eq'n has a unique sol'n, namely $c_1 = \cdots = c_m = 0$

5.7

Comment 2:

If the vectors $c_1u_1, \dots, c_mu_m$ can be pictured as arrowed entities then

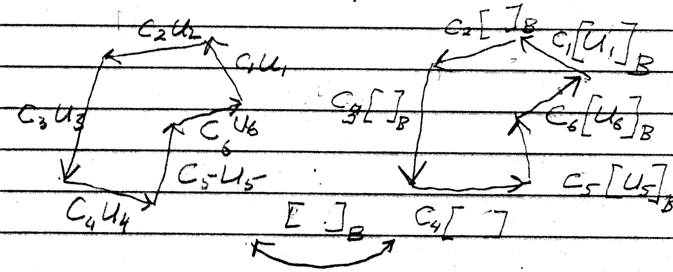


Figure 5.7

Closed m-sided polygons (in V) get mapped

$\Leftrightarrow$ closed m-sided polygons in R^p.

Deferred to Lect. 6

P6.4

5.8 ~~4.7~~

Comment:

The question of linear (in)dependence has been reduced to an algebraic problem about the uniqueness of a solution to the system Eqs (★★) on P5.6

Example!: $\{1, x, x^2\} = B$ is a basis for P_2.

SHOW ~~that~~ $\{1, x+1, (x+1)^2\}$ is a linearly independent set.

$$[1]_B = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad [x+1]_B = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad [(x+1)^2]_B = \begin{bmatrix}1\\2\\1\end{bmatrix} \in R^3$$

We ~~note that~~ $p \equiv c_1 + (x+1)c_2 + (x+1)^2 c_3 = \vec{0} \Leftrightarrow$

$$\begin{bmatrix}1&1&1\\0&1&2\\0&0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

2nd way!

Letting $x=-1 \Rightarrow c_1 = 0$

Letting $x = 0 \Rightarrow c_2 + c_3 = 0$

" $x = -2 \Rightarrow -c_2 + c_3 = 0$

$\Downarrow$

$c_2 = c_3 = 0$

has only the trivial solution

$$[p]_B \equiv \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Corollary to Theorem 5 ("Too many vectors $\Rightarrow$ lin. dep.")

If $B = \{v_1, \dots v_p\}$ is a basis for V, then any set of $p+1$ vectors is linearly dependent.

proof: Let $\{u_1, u_2, \dots, u_{p+1}\}$ be a collection of $p+1$ vectors. Use Theorem 5 to obtain p homogeneous equations in $p+1$ unknowns. This always has a nontrivial solution. QED.

On The Psycho-epistemological Roots of Efficient Thinking.

It is easy to point to products of human thought, such as concepts, generalizations, principles, theories, philosophies etc. Why are some person's good at this production process while others are not?

Why do some person's come up with good products while others come up with worthless ones or even with bad ones?

These are psycho-epistemological questions, meaning that they pertain to the nature of one's thinking process.

Thinking is the process of one's conscious mind interacting with one's subconscious, the "hard drive" of your consciousness the hard drive whose content one programs and retrieves.

The quality of the products of one thinking depends, (among others) on the content of one's subconscious: If it consists of invalid or hazily defined concepts, then so will be one's thinking that uses them. If the concepts are valid and clearly defined, then this will be reflected in one's thinking based on such ideas.

If one programs one's subconscious with mental fluff and ideas disconnected from the real world, then this reflects on the products of one's thinking. On the other hand, if one programs one's subconscious, (one's "hard drive") with consciously chosen non-contradictory ideas that integrate with what is there already, then this reflects on one's thinking accordingly.

Such a circumstance can be summarized by saying: "Garbage In, Garbage Out" (G.I.G.O), or stated positively:

" Valid ideas presuppose valid concepts