LECTURE 5

1. Basis-induced correspondence between V and R^p as structure preserving
2. Preservation of linear independence and linear dependence of sets of vectors
3. Isomorphism and its basis indendence.

The implicit basis-induced one-to-one correspondence between V and R^p, which is expressed by Theorem 4 on page 4.3, is the doorway between (i) the highly abstract (and hence widely applicable) <u>geometric approach</u> to grasping the nature of the world and (ii) the <u>computational approach</u> (i.e. via the linear algebra of R^p) which concretizes the nature of the world in always the same way, namely, by means of numbers as is done with simulation software (e.g. MATLAB) on a computer.

The choice of a basis for a vector space establishes the connecting link between the elements of an abstract vector space and the concrete directly measured results, elements of R^p.

"Isomorphism" Between V and R^p

Theorem 4 says:

Given a basis $B = \{v_1, \dots, v_p\} \subset V$, for any vector $v \in V$

there exists uniquely defined scalars $\{a_1, \dots, a_p\}$ such that

$$w = a_1 v_1 + \dots + a_p v_p$$

given the basis B

Thus for any $w \in V$ $\exists$ a p-tuple $\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}_B$:

$$w \xrightarrow{[\ ]_B} \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}_B \equiv [w]_B$$

$$V \xrightarrow{[\ ]_B} R^p$$

For the given basis this p-tuple is unique implies

$$\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}_B \xrightarrow{[\ ]_B^{-1}} a_1 v_1 + \dots + a_p v_p \equiv w \leftarrow \text{(unique)}$$

$$R^p \xrightarrow{[\ ]_B^{-1}} V$$

Note the following important feature

$$\boxed{\left.\begin{array}{l}\text{Existence} \Leftrightarrow \text{spanning property} \\ \text{Uniqueness} \Leftrightarrow \text{linear independence}\end{array}\right\} \text{of } B.}$$

Thus one has the following

<u>Representation</u> 5.4

<u>Theorem</u> ($V \leftrightarrow R^p$ is structure preserving)

a) A basis $B = \{v_1, \dots, v_p\} \subset V$ for vector space V

induces the 1-1 mapping $[\ ]_B$:

$$V \overset{[\ ]_B}{\longleftrightarrow} R^p$$

$$w = a_1 v_1 + \dots + a_p v_p \longleftrightarrow [w]_B \equiv \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix},$$

namely for every w there is a unique $[w]_B$ <u>and</u> for every $[w]_B$ there is a unique w:

$$w = a_1 v_1 + \dots + a_p v_p.$$

b) This mapping preserves structure,

namely

(i) addition

(ii) scalar multiplication

(iii) linear combinations

(iv) linear independence

i.e. one has

$$\left.\begin{matrix}(i)\\(ii)\\(iii)\end{matrix}\right\} \boxed{[c_1 w_1 + c_2 w_2]_B = c_1[w_1]_B + c_2[w_2]_B} \quad (*)$$

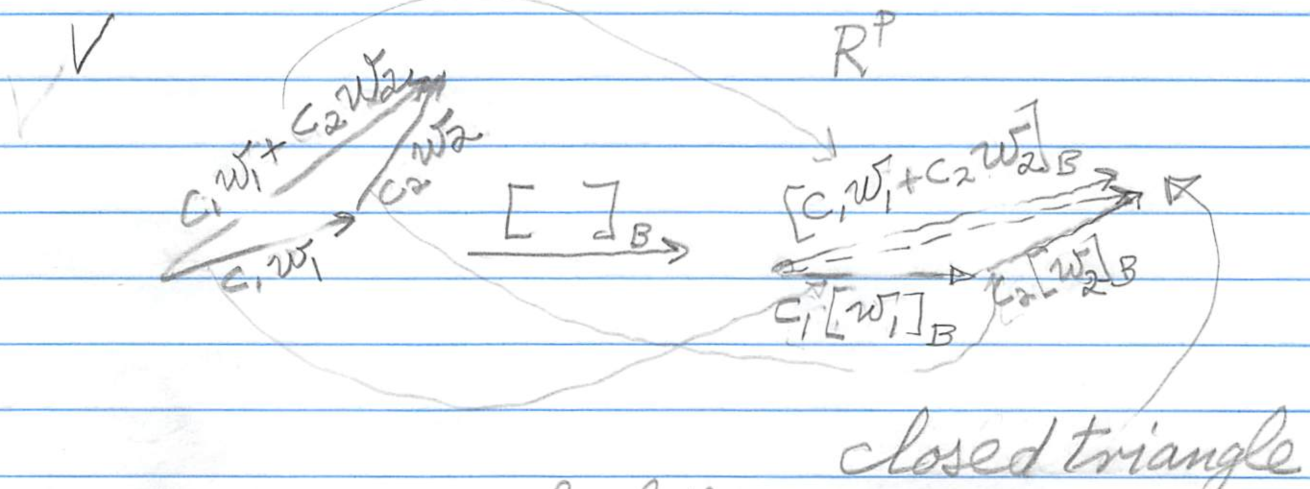
(iv)} $\{w_1, \dots, w_k\}$ is linearly independent (dependent)

$\Leftrightarrow \{[w_1]_B, \dots, [w_k]_B\}$ is lin. indep. (depend.)

Comment:

Q: What is the geometrical meaning of Eq. (*)?

A: The correspondence $[\ ]_B$ maps closed triangles in V to closed triangles in R^p:



The algebraic proof of this geometrical fact is as follows:

1. Let $\{v_1, \ldots, v_p\}$ be a basis for V then

$$[c_1 w_1 + c_2 w_2]_B = \left[c_1(a_1 v_1 + \cdots + a_p v_p) + c_2(b_1 v_1 + \cdots + b_p v_p)\right]_B$$

$$= \left[(c_1 a_1 + c_2 b_1) v_1 + \cdots + (c_1 a_p + c_2 b_p) v_p\right]_B$$

(**)

$$= \begin{bmatrix} c_1 a_1 + c_2 b_1 \\ \vdots \\ c_1 a_p + c_2 b_p \end{bmatrix}_B = c_1 \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}_B + c_2 \begin{bmatrix} b_1 \\ \vdots \\ b_p \end{bmatrix}_B = c_1 [w_1]_B + c_2 [w_2]_B$$

Q.E.D.

Preservation of linear structure by a basis-induced representation means that all vector space operations in V are mirrored in R^p. This means that all computations in V can be achieved by doing them in R^p

These computation are done in terms of matrices.

Computations involving linear (in)dependence illustrate this principle

Theorem 5 (Preservation of linear (in)dependence)

Let $B = \{v_1, \ldots, v_p\}$ be a basis for V

Then

$\{u_1, \ldots, u_m\}$ is lin. indep in $V \Leftrightarrow \{[u_1]_B, \ldots, [u_m]_B\}$ is lin. indep. in R^p.

WHY? Know why this is so.

Comment 1: This conclusion is equivalent to:

$\{u_1, \ldots, u_m\}$ is lin. dep in $V \Leftrightarrow \{[u_1]_B, \ldots, [u_m]_B\}$ is lin. dep in R^p.

Comment 2: This theorem is validated by starting in V, but doing the computation in R^p.

proof: 1) Consider the following vectorial equation in V:

$$u_1 c_1 + u_2 c_2 + \cdots + u_m c_m = \vec{0} \quad (\star)$$

2) In R^p we have

by Eq.(*) on P5.3

$$[u_1]_B c_1 + \cdots + [u_m]_B c_m = [\vec{0}]_B$$

by Eq.(**) on P5.4

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{p1} \end{bmatrix}_B c_1 + \cdots + \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{pm} \end{bmatrix}_B c_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_B$$

or

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & & \vdots \\ a_{p1} & \cdots & a_{pm} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_B \quad (\star\star)$$

This is p equations in m unknowns, where p = # of basis vectors

Comment: If the columns of the matrix form an independent set, then $c_1 = \cdots = c_m = 0$ is the only solution to $(\star)$.

$\boxed{\Leftarrow \text{ of Thm5}}$

If the columns of the matrix form a dependent set then $\exists$ a non-trivial solution $[c_1, \ldots, c_m]^t$ to Eq.$(\star)$.

$\boxed{\Rightarrow \text{ of Thm5}}$

Conclusion:

$\{u_1, \ldots, u_m\}$ forms a l. indep. set $\Leftrightarrow$ the eq'n has a unique sol'n, namely $c_1 = \cdots = c_m = 0$

Comment 2:

If the vectors $c_1u_1, \ldots, c_m u_m$ can be pictured as arrowed entities then

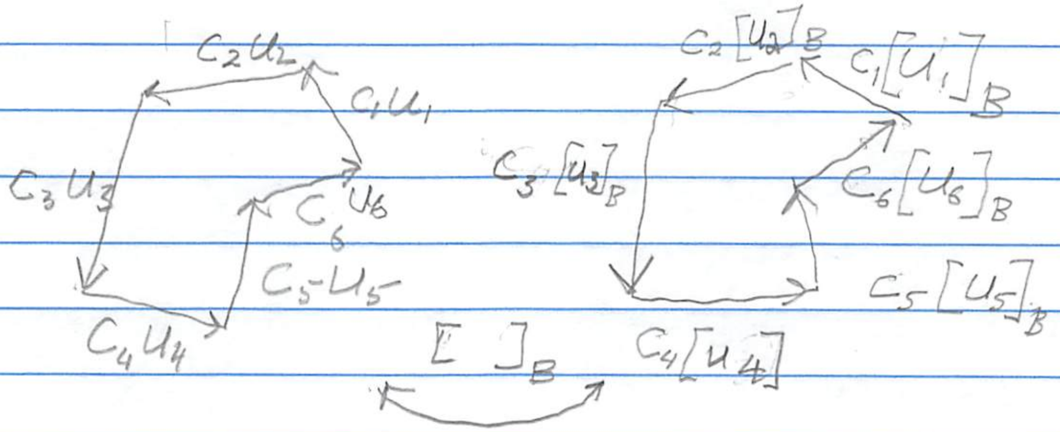


Figure 5.7

Closed m-sided polygons (in V) get mapped

$\Leftrightarrow$ closed m-sided polygons in R^p.