LECTURE 5

1. Basis-induced correspondence between $V$ and $\mathbb{R}^n$ as structure preserving.

2. Preservation of linear independence and linear dependence of sets of vectors.

3. Isomorphism and its basis independence.
The implicit basis-induced one-to-one correspondence between $V$ and $\mathbb{R}^p$ which is expressed by Theorem 4 on page 43 is the doorway between (i) the highly abstract (and hence widely applicable) geometric approach to grasping the nature of the world and (ii) the computational approach (i.e. via the linear algebra of $\mathbb{R}^p$) which concretizes the nature of the world in always the same way, namely by means of numbers as is done with simulation software (e.g. MATLAB) on a computer.
The choice of a basis for a vector space establishes the connecting link between the elements of an abstract vector space and the concrete directly measured results, elements of $\mathbb{R}^n$. 
"Isomorphism" Between $V$ and $\mathbb{R}^p$

Theorem 4 says:

Given a basis $B = \{v_1, \ldots, v_p\} = V$, for any vector $v \in V$, there exists uniquely defined scalars $a_1, \ldots, a_p$ such that

$$w = a_1 v_1 + \cdots + a_p v_p$$

given the basis $B$

Thus for any $w \in V$, $\exists$ a $p$-tuple $[a_1] = \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}$ such that

$$w \equiv [w]_B \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix}$$

$V \leftarrow J^B \rightarrow \mathbb{R}^p$

For the given basis

This $p$-tuple is unique implies

$$\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix} a_1, \ldots, a_p \quad \equiv \quad w \rightarrow (unique)$$

$\mathbb{R}^p \leftarrow J^B \rightarrow V$

Note the following important feature

Existence $\iff$ spanning property of $B$. 

Uniqueness $\iff$ linear independence of $B$. 

Thus one has the following representation formula:

**Theorem:** \( \varphi : V \rightarrow \mathbb{R}^p \) is structure preserving.

(a) A basis \( B = \{ v_1, \ldots, v_p \} \subseteq V \) for vector space \( V \) induces the \( 1 \rightarrow 1 \) mapping \( \varphi_B : V \rightarrow \mathbb{R}^p \):

\[
\begin{align*}
V & \rightarrow \mathbb{R}^p \\
[a_1] & \rightarrow [w]^B = \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix},
\end{align*}
\]

namely for every \( w \) there is a unique \( [w]^B \) and for every \( [w]^B \) there is a unique \( w \):

\[
w = a_1 v_1 + \cdots + a_p v_p.
\]

(b) This mapping preserves structure, namely

- (i) addition
- (ii) scalar multiplication
- (iii) linear combinations
- (iv) linear independence

i.e., one has (iii):

\[
[C_1 w_1 + C_2 w_2]^B = C_1 [w_1]^B + C_2 [w_2]^B \quad (\ast)
\]

(i) \( \{ w_1, \ldots, w_k \} \) is linearly independent (dependent)

\( \iff \) \( \{ [w_1]^B, \ldots, [w_k]^B \} \) is lin. indep. (depend.)
Comment:

Q: What is the geometrical meaning of Eq. (4)?

A: The correspondence \[ J_B \] maps closed triangles in \( V \) to closed triangles in \( \mathbb{R}^p \):

![Diagram showing the mapping]

The algebraic proof of this geometrical fact is as follows:

1. Let \( \{ e_1, \ldots, e_p \} \) be a basis for \( V \) then

\[
\begin{bmatrix}
C_{1,1} + C_{2,2} f_2
\end{bmatrix}_B = \begin{bmatrix}
C_1 \begin{bmatrix}
0 & \ldots & 0 & a_2 e_2
\end{bmatrix} + C_2 \begin{bmatrix}
0 & \ldots & 0 & b_2 e_2
\end{bmatrix}
\end{bmatrix}_B
\]

\[
= \begin{bmatrix}
C_1 a_1 + C_2 b_2
\end{bmatrix}_B + \begin{bmatrix}
\vdots & \ddots & \ddots & \vdots
0 & \ldots & 0 & C_2 b_p
\end{bmatrix}_B
\begin{bmatrix}
C_1 a_1 + C_2 b_2
\end{bmatrix}_B + C_2 \begin{bmatrix}
C_1 \begin{bmatrix}
0 & \ldots & 0 & a_p e_p
\end{bmatrix} + C_2 \begin{bmatrix}
0 & \ldots & 0 & b_p e_p
\end{bmatrix}
\end{bmatrix}_B
\]

\[
= C_1 \begin{bmatrix}
a_1
\end{bmatrix}_B + C_2 \begin{bmatrix}
b_1
\end{bmatrix}_B + \begin{bmatrix}
0 & \ldots & 0 & C_2 b_p
\end{bmatrix}_B
\begin{bmatrix}
C_1 a_1 + C_2 b_2
\end{bmatrix}_B + C_2 \begin{bmatrix}
C_1 \begin{bmatrix}
0 & \ldots & 0 & a_p e_p
\end{bmatrix} + C_2 \begin{bmatrix}
0 & \ldots & 0 & b_p e_p
\end{bmatrix}
\end{bmatrix}_B
\]

\[
= C_1 \begin{bmatrix}
a_1
\end{bmatrix}_B + C_2 \begin{bmatrix}
b_1
\end{bmatrix}_B + \begin{bmatrix}
0 & \ldots & 0 & C_2 b_p
\end{bmatrix}_B
\begin{bmatrix}
C_1 a_1 + C_2 b_2
\end{bmatrix}_B + C_2 \begin{bmatrix}
C_1 \begin{bmatrix}
0 & \ldots & 0 & a_p e_p
\end{bmatrix} + C_2 \begin{bmatrix}
0 & \ldots & 0 & b_p e_p
\end{bmatrix}
\end{bmatrix}_B
\]

\[
\Rightarrow \text{ED.}
\]
Preservation of linear structure by a basis-induced representation means that all vector space operations in \( V \) are mirrored in \( \mathbb{R}^p \). This means that all computations in \( V \) can be achieved by doing them in \( \mathbb{R}^p \).

These computations are done in terms of matrices.

Computations in solving linear (in)dependence illustrate this principle.

**Theorem 5 (Preservation of linear (in)dependence)**

Let \( B = \{ v_1, \ldots, v_p \} \) be a basis for \( V \). Then

\[ \{ u_1, \ldots, u_m \} \text{ is lin. indep in } V \iff \{ [u_1]_B, \ldots, [u_m]_B \} \]

is lin. indep in \( \mathbb{R}^p \).

*Comment 1:* This conclusion is equivalent to:

\[ \{ u_1, \ldots, u_m \} \text{ is lin. dep in } V \iff \{ [u_1]_B, \ldots, [u_m]_B \} \]

is lin. dep in \( \mathbb{R}^p \).

*Comment 2:* This theorem is validated by starting in \( V \), but doing the computation in \( \mathbb{R}^p \).
Comment: If the columns of the matrix form an independent set then \( \mathbb{E} \) is a non-trivial equation system. This is \( p \) equations on \( n \) unknowns where \( p \approx n \).

Consequence: If the columns of the matrix form an independent set then \( \mathbb{E} \) has a unique solution, namely \( x = 0 \) and \( \mathbb{E} \) is in row echelon form.

Proof: 1) Consider the following vectorial equation:

\[
\begin{bmatrix}
-1 & 2 & 0 & \cdots & 0 \\
0 & -1 & 2 & \cdots & 0 \\
0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix}
= \begin{bmatrix} 0 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

by \( Eq.(1) \) or \( Eq.(2) \) on \( \mathbb{R}^n \).
Comment 2:

If the vectors $u_1, \ldots, u_m$ can be pictured as arrowed entities then

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Figure 5.7

Closed $m$-sided polygons get mapped

$\leftrightarrow$ Closed $m$-sided polygons in $\mathbb{R}^p$