LECTURE 6

Wednesday

1. Structure preservation (again)
2. Dimension of V: its coordinate invariance.
3. Mutual implication of linear dependence and independence. (in an appropriate context)

Structure Preservation (A recap)

The Representation Theorem (page 5.3) and Theorem 5 (page 5.5) read as follows:

A chosen basis, say

$$B = \{v_1, \dots, v_p\}$$

for a vector space V induces a mapping

$$V \longleftrightarrow R^p$$

which preserves the vector space structure by which one means:

Given: $\{u_1, \dots, u_m\} \subset V$

Then

$$(*) \quad u_1 c_1 + \dots + u_m c_m = \vec{0} \text{ has } c_1 = \dots = c_m = 0$$

as its only solution

i.e. $\{u_1, \dots, u_m\}$ is lin. indep.

$\Longleftrightarrow$

$$(**) \quad \begin{array}{ccc} \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{p1} & & a_{pm} \end{bmatrix} & \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = & \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \text{ has } c_1 = \dots = 0 \text{ as its only sol'n}$$

$$\underset{[u_1]_B}{\sim} \quad \underset{[u_m]_B}{\sim} \quad \underset{[\vec{0}]_B}{\sim}$$

i.e. $\{[u_1]_B, \dots, [u_m]_B\}$ is a lin. indep. set

~~proof~~ by contradiction:

$\Longleftarrow$: assume (*) has a non-zero sol'n $\{c_1, \dots, c_m\}$

Then by Theorem 4 on page 4.3 one has

$$[(*)]_B = [0]_B \Rightarrow [u_1 c_1 + \dots + u_m c_m]_B = [\vec{0}]_B$$

$$[u_1]_B c_1 + \dots + [u_m]_B c_m = [\vec{0}]_B$$

which contradict the fact that Eq. (**) has only $c_1 = \dots = c_m = 0$ as its solution.

proof by contradiction:

$\Longrightarrow$: assume (**) has a non-zero solution,

We have

$$[u_1]_B c_1 + \dots + [u_m]_B c_m = [0]_B$$

or

$$a_{11} c_1 + \dots + a_{1m} c_m = 0 \quad \Big| \times v_1$$

$$a_{p1} c_1 + \dots + a_{pm} c_m = 0 \quad \Big| \times v_p$$

Mply respective rows by $v_1, \dots, v_p$ respectively.

Add, rearrange and obtain

$$u_1 c_1 + \dots + u_m c_m = \vec{0}$$

where some of the c_i's are non-zero.

This contradicts the given information about Eq. (*). Hence the assumption is wrong.

Q.E.D.

Comment 1:

The equivalence relation "$\Leftrightarrow$" between Eqs(*) and (**) also holds if one replaces "lin indep" with "lin dep."

Comment 2:

The question of linear (in)dependence in V has been reduced to an algebraic problem about the uniqueness of a solution to the system of Eqs (**) on page 6.1

Example 1

Given $\{1, x, x^2\} = B$ is a basis for P_2

Show that $\{1, 1+x, (1+x)^2\}$ is a linearly independent set.

There are two ways of showing this.

First way (Do the calculation in R^3)

$$[1]_B = \begin{bmatrix}1\\0\\0\end{bmatrix}, [1+x]_B = \begin{bmatrix}1\\1\\0\end{bmatrix}, [(1+x)^2]_B = \begin{bmatrix}1\\2\\1\end{bmatrix} \in R^3.$$

Note that the equation

$p \equiv c_1 + (1+x)c_2 + (1+x)^2 c_3 = \vec{0}$ implies and is implied by

$$\begin{bmatrix}1&1&1\\0&1&2\\0&0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

and has only the trivial solution

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

<u>Second way</u> (Do the calculation in $\mathbb{P}_2$) 6.5

Consider $p(x) \equiv 1c_1 + (1+x)c_2 + (1+x)^2 c_3 = \vec{0}$ (*) $\forall x$

Evaluate $p(x)$ at various x values

$p(-1) \Rightarrow c_1 + 0 \cdot c_2 + 0 \cdot c_3 = 0 \Rightarrow c_1 = 0$

$$\left.\begin{array}{l} p(0) \Rightarrow c_2 + c_3 = 0 \\ p(-2) \Rightarrow -c_2 + c_3 = 0 \end{array}\right\} \Rightarrow c_2 = c_3 = 0$$

Thus $c_1 = c_2 = c_3 = 0$ is the only solution to (*).

Thus $\{1, 1+x, (1+x)^2\}$ is lin. indep. indeed.

<u>Corollary to Theorem 5</u> (page 5.6)
("Too many vectors $\Rightarrow$ lin. dependence")

Let $B = \{v_1, \dots, v_p\}$ a basis for V.

Conclusion:

Any set of $p+1$ vectors is a linearly dependent set

<u>proof: Doing the calculation in R^p one proceeds as follows: Let</u>
$\{u_1, \dots, u_p, u_{p+1}\}$ be a collection of $p+1$ vectors. Consider $\vec{u}_1 c_1 + \cdots + \vec{u}_p c_p + \vec{u}_{p+1} c_{p+1} = \vec{0}$.

Take the coordinate representative of both sides of this equation.

One obtains a system of p homogeneous equations in $p+1$ unknowns. Such a system always has a non-trivial solution. Hence $\{u_1, \dots, u_p, u_{p+1}\}$ is a lin. dep. set.

<u>Comment</u>: This corollary is useful for showing that, regardless of one's choice of basis for a vector space, the number of basis elements is always the same. This is stated by the following

Second way (Do the calculation in $\mathbb{P}_2$)

Consider $p(x) = 1c_1 + (1+x)c_2 + (1+x)^2 c_3 = \vec{0}$ (*) $\forall x$

Evaluate $p(x)$ at various x values

$$p(-1) \Rightarrow c_1 + 0 \cdot c_2 + 0 \cdot c_3 = 0 \Rightarrow c_1 = 0$$

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Corollary to Theorem 5 (page 5.5)
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Conclusion:

Any set of $p+1$ vectors is a linearly dependent set

proof: Doing the calculation in R^p, one proceeds as follows: Let

$$\{u_1, \ldots, u_p, u_{p+1}\}$$

be a collection of $p+1$ vectors. Take the coordinate representative of Eq. (*) on Page 6.1 with $m = p+1$, ($u_1c_1 + \cdots + u_pc_p + u_{p+1}c_{p+1} = \vec{0}$). One obtains a system of p homogeneous equations in $p+1$ unknowns. Such a system always has a non-trivial solution.

Comment: This corollary is useful for showing that, regardless of one's choice of basis for a vector space, the number is of basis elements is always the same. This is stated by the following

Theorem 6

Let $B=\{v_1, \ldots, v_p\}$ be a basis for V

Let $Q=\{u_1, \ldots, u_m\}$ be another basis for V

Conclusion: $m=p$.

Proof: The validation uses the corollary on page 6.5 as follows:

(i) There are only two mutually exclusive possibilities

$\boxed{\text{either}}$ $\underset{(a)}{m\le p}$ $\boxed{\text{or}}$ $\underset{(b)}{m>p}$

The corollary on p. 6.5 rules out (b) because Q is linearly independent. Consequently, we are left with (a),

$$\boxed{m\le p}$$

(ii) Similary for

$$m<p \text{ or } m\ge p$$

one has to rule out $m<p$ because B is a basis. Hence

$$\boxed{m\ge p}$$

(iii) Combine (i) and (ii) to obtain

$$\boxed{m=p}$$

Definition 6 (Dimension of a vector space)

a) If V has a basis $B=\{v_1, \ldots, v_n\}$ of n vectors then V is said to have dimension n; or

$$\dim V = n$$

Comment: n is a basis independent concept.

b) If V does not have a finite basis, then V is said to be infinite dimensional.

<u>Comment</u>

The concept of dimension is useful because, among others, linear independence and the spanning property are mutually implied properties for certain sets of vectors.

<u>Theorem 7</u> (Independence $\Leftrightarrow$ spanning ppty)

$\dim V = p \Rightarrow$ a) Any lin. independent set of p vectors, say $\{u_1, \ldots, u_p\} \equiv Q$, is a spanning set for V

$\Rightarrow$ b) Any spanning set $\{u_1, \ldots, u_p\} \equiv Q$ is linearly indep.

Example 2. (Cont'd from Example 1 on p. 6.4)

$\dim P_2 = 3$ because $B = \{1, x, x^2\}$

$Q = \{1, 1+x, (1+x)^2\}$ is linearly independent.

$\therefore$ $Sp(Q) = P_2$, i.e. Q is a spanning set for P_2

Proof of Theorem 7 on p. 6.9

a) Let $v \in V$. Then $\{v, u_1, \ldots, u_p\}$ is lin. dep.,

i.e. $v c_0 + u_1 c_1 + \cdots + u_p c_p = \vec{0}$

has a non-trivial solution with $c_0 \neq 0$ because $\{u_1, \ldots, u_p\}$ being lin. indep would make $c_1 = \cdots = c_p = 0$. Thus

$$v = \frac{-1}{c_0}[u_1 c_1 + \cdots + u_p c_p]$$

Hence $\{u_1, \ldots, u_p\}$ spans V indeed.

b) proof by contradiction:

Assume $\{u_1, \ldots, u_p\}$ is a spanning set which is linearly <u>dependent</u>. Thus

$$u_1 c_1 + \cdots + u_{p-1} c_{p-1} + u_p c_p = 0$$

has a non-trivial solution.

Without loss of generality one can assume that $c_p \neq 0$ so that one can solve for u_p, i.e. $u_p \in Sp(\{u_1, \ldots, u_{p-1}\})$. Continuing this way one arrives finally at a linearly independent spanning set

$$\{u_1, \ldots, u_m\} \quad \boxed{m < p},$$

i.e. an m-dimensional basis for V. This contradicts Theorem 6, which requires $m = p$. Consequently, our original assumption about Q was false.

Hence

$$Q = \{u_1, \ldots, u_p\} \text{ is}$$

a linearly independent indeed.