LECTURE 6 (Wednesday)

0. Prologemenon to Engineering and Science.

1. Structure preservation (again)

2. Dimension of V: its coordinate invariance

3. Mutual implication of linear dependence and independence (in an appropriate context)

On The Psycho-epistemological Roots of Efficient Thinking.

\Begin: It is easy to point to products of human thought, such as concepts, generalizations, principles, theories, philosophies, etc. Why are some persons good at this production process while others are not?

Why do some persons come up with good products while others come up with worthless ones or even with bad ones?

These are psycho-epistemological questions, meaning that they pertain to the nature of one's thinking process.

Thinking is the process of one's conscious mind interacting with one's subconscious, the "hard drive" of your consciousness, the hard drive whose content one programs and retrieves.

The quality of the products of one thinking depends, (among others), on the content of one's subconscious: If it consists of invalid or hazily defined concepts, then so will be one's thinking that uses them. If the concepts are valid and clearly defined, then this will be reflected in one's thinking based on such ideas.

If one programs one's subconscious with mental fluff and ideas disconnected from the real world, then this reflects on the products of one's thinking. On the other hand, if one programs one's subconscious, (one's "hard drive") with consciously chosen non-contradictory ideas that integrate with what is there already, then this reflects on one's thinking accordingly.

Such a circumstance can be summarized by saying: "Garbage In, Garbage Out" (G.I.G.O), or stated positively:

"Valid ideas presuppose valid concepts."

V. End.

Structure Preservation (A recap)

The Representation Theorem (page 5.3) and Theorem 5 (page 5.5) read as follows:

A chosen basis, say,

$$B = \{v_1, \ldots, v_p\}$$

for a vector space V induces a mapping

$$V \longleftrightarrow R^p$$

which <u>preserves</u> the <u>vector space structure</u> by which one means:

Given: $\{u_1, \ldots, u_m\} \subset V$

Then

$$(*) \quad u_1 c_1 + \cdots + u_m c_m = \vec{0} \text{ has } c_1 = \cdots = c_m = 0$$

as its only solution

i.e. $\{u_1, \ldots, u_m\}$ is <u>lin. indep.</u>

$\Longleftrightarrow$

$$(**) \quad \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{p1} & & a_{pm} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \text{ has } c_1 = \cdots = 0 \text{ as its only sol'n}$$

$$\underbrace{\phantom{a_{11}}}_{[u_1]_B} \quad \underbrace{\phantom{a_{1m}}}_{[u_m]_B} \quad \underbrace{}_{[\vec{0}]_B}$$

i.e. $\{[u_1]_B, \ldots, [u_m]_B\}$ is a <u>lin. indep.</u> set

~~proof~~ by contradiction:

$\Leftarrow$: assume (*) has a non-zero sol'n $\{c_1, \ldots, c_m\}$, i.e. $\{u_1, \ldots, u_m\}$ is lin. dep.

Then by Theorem 4 on page 4.3 one has

$$[(*)]_B = [0]_B \Rightarrow [u_1 c_1 + \cdots + u_m c_m]_B = [\vec{0}]_B$$

$$[u_1]_B c_1 + \cdots + [u_m]_B c_m = [\vec{0}]_B,$$

which contradict the fact that Eq. (**) has only $c_1 = \cdots = c_m = 0$ as its solution.

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proof by contradiction:

$\Longrightarrow$: assume (**) has a non-zero solution, i.e the columns of $[a_{ij}]$ form a lin. dep. set.

i.e. we have $[u_1]_B c_1 + \cdots + [u_m]_B c_m = [0]_B$

where some of the c_j's are non-zero.

We have

$$\left.\begin{aligned} a_{11} c_1 + \cdots + a_{1m} c_m &= 0 \\ &\vdots \\ a_{p1} c_1 + \cdots + a_{pm} c_m &= 0 \end{aligned}\right| \begin{aligned} &\times v_1 \\ &\vdots \\ &\times v_p \end{aligned}$$

Mply respective rows by $v_1, \ldots, v_p$ respectively.

Add, rearrange and obtain

$$u_1 c_1 + \cdots + u_m c_m = \vec{0}$$

where some of the c_i's are non-zero.

This contradicts the given information about Eq. (*). Hence the assumption is wrong.

Q.E.D.

Comment 1:

The equivalence relation "$\Leftrightarrow$" between Eqs(*) and (**) also holds if one replaces "lin. indep." with "lin. dep."

Comment 2:

The question of linear (in)dependence in V has been reduced to an algebraic problem about the uniqueness of a solution to the system of Eqs(**) on page 6.1

Example 1

Given $\{1, x, x^2\} = B$ is a basis for P_2

Show that $\{1, 1+x, (1+x)^2\}$ is a linearly independent set.

There are **two ways** of showing this.

First way. (Do the calculation in R^3)

$$[1]_B = \begin{bmatrix}1\\0\\0\end{bmatrix}, [1+x]_B = \begin{bmatrix}1\\1\\0\end{bmatrix}, [(1+x)^2]_B = \begin{bmatrix}1\\2\\1\end{bmatrix} \in R^3.$$

Note that the equation

$$p \equiv c_1 + (1+x)c_2 + (1+x)^2 c_3 = \vec{0}$$ implies and is implied by

$$\begin{bmatrix}1&1&1\\0&1&2\\0&0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

and has only the trivial solution

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

<u>Second way</u> (Do the calculation in $\mathbb{P}_2$)

Consider $p(x) = 1c_1 + (1+x)c_2 + (1+x)^2 c_3 = \vec{0}$ (*) $\forall x$

Evaluate $p(x)$ at various x values

$$p(-1) \Rightarrow c_1 + 0 \cdot c_2 + 0 \cdot c_3 = 0 \Rightarrow c_1 = 0$$

$$\left.\begin{aligned} p(0) &\Rightarrow c_2 + c_3 = 0 \\ p(-2) &\Rightarrow -c_2 + c_3 = 0 \end{aligned}\right\} \Rightarrow c_2 = c_3 = 0$$

Thus $c_1 = c_2 = c_3 = 0$ is the only solution to (*).

Thus $\{1, 1+x, (1+x)^2\}$ is lin. indep. indeed.

<u>Corollary to Theorem 5</u> (page 5.6)

("Too many vectors $\Rightarrow$ lin. dependence")

Let $B = \{v_1, \ldots, v_p\}$ a basis for V.

Conclusion:

Any set of $p+1$ vectors is a linearly dependent set proof: Doing the calculation in R^p, one proceeds as follows: Let $\{u_1, \dots, u_p, u_{p+1}\}$ be a collection of $p+1$ vectors. Consider $\vec{u}_1 c_1 + \dots + \vec{u}_p c_p + \vec{u}_{p+1} c_{p+1} = \vec{0}$.

Take the coordinate representative of both sides of this equation.

One obtains a system of p homogeneous equations in $p+1$ unknowns. Such a system always has a non-trivial solution. Hence $\{u_1, \dots, u_p, u_{p+1}\}$ is a lin. dep. set.

Comment: This corollary is useful for showing that, regardless of one's choice of basis for a vector space, the number of basis elements is always the same. This is stated by the following

Theorem 6

Let $B = \{v_1, \dots, v_p\}$ be a basis for V

Let $Q = \{u_1, \dots, u_m\}$ be another basis for V

Conclusion: $m = p$.

Proof: The validation uses the corollary

on page 6.5 as follows:

(i) There are only two mutually exclusive possibilities

$\boxed{\text{either}}$ $m \le p$ (a) $\boxed{\text{or}}$ $m > p$ (b)

The corollary on p6.5 rules out (b)

because Q is linearly independent.

Consequently, we are left with (a),

$$\boxed{m \le p}$$

(ii) Similary for

$$m < p \quad \text{or} \quad m \geq p$$

one has to rule out $m < p$ because B is lin. indep

$$\boxed{m \geq p.}$$

(iii) Combine (i) and (ii) to obtain

$$\boxed{m = p}$$

<u>Definition 6</u> (Dimension of a vector space)

a) If V has a basis $B = \{v_1, \ldots, v_n\}$ of n vectors then V is said to have <u>dimension</u> n; or

$$\dim V = n$$

<u>Comment</u>: n is a basis independent concept,

b) If V does not have a finite basis, then V is said to be <u>infinite dimensional</u>.

Comment

The concept of dimension is useful, because, among others, linear independence and the spanning property are mutually implied properties for certain sets of vectors.

Theorem 7 (Independence $\Leftrightarrow$ spanning ppty)

$\dim V = p \Rightarrow$ a) Any lin. independent set of p vectors, say $\{u_1, \ldots, u_p\} \equiv Q$, is a spanning set for V

$\Rightarrow$ b) Any spanning set $\{u_1, \ldots, u_p\} \equiv Q$ is linearly indep.

Example 2. (Cont'd from Example 1 on p. 6.4)

dim $\mathbb{P}_2 = 3$ because $\mathcal{B} = \{1, x, x^2\}$

$Q = \{1, 1+x, (1+x)^2\}$ is linearly independent.

$\therefore$ $Sp(Q) = \mathbb{P}_2$, i.e. Q is a spanning set for $\mathbb{P}_2$.

Proof of Theorem 7 on P 6.9

a) Let $v \in V$. Then $\{v, u_1, \ldots, u_p\}$ is lin. dep.,

i.e. $v c_0 + u_1 c_1 + \cdots + u_p c_p = \vec{0}$

has a non-trivial solution with $c_0 \neq 0$, because $\{u_1, \ldots, u_p\}$ being lin. indep. would make $c_1 = \cdots = c_p = 0$. Thus

$$v = \frac{-1}{c_0}\left[u_1 c_1 + \cdots + u_p c_p\right]$$

Hence $\{u_1, \ldots, u_p\}$ spans V indeed.

b) proof by contradiction:

Assume $Q = \{u_1, \ldots, u_p\}$ is a spanning set which is linearly <u>dependent</u>. Thus

$$u_1 c_1 + \cdots + u_{p-1} c_{p-1} + u_p c_p = 0$$

has a non-trivial solution.

Without loss of generality one can assume that $c_p \neq 0$ so that one can solve for u_p, i.e. $u_p \in Sp(\{u_1, \ldots, u_{p-1}\})$. Continuing this way one arrives finally at a linearly independent spanning set

$$\{u_1, \ldots, u_m\} \quad \boxed{m < p},$$

i.e. an m-dimensional basis for V. This contradicts Theorem 6, which requires $m = p$. Consequently, our original assumption about Q was false. ∴

Hence

$$Q = \{u_1, \ldots, u_p\} \text{ is}$$

a linearly independent indeed.