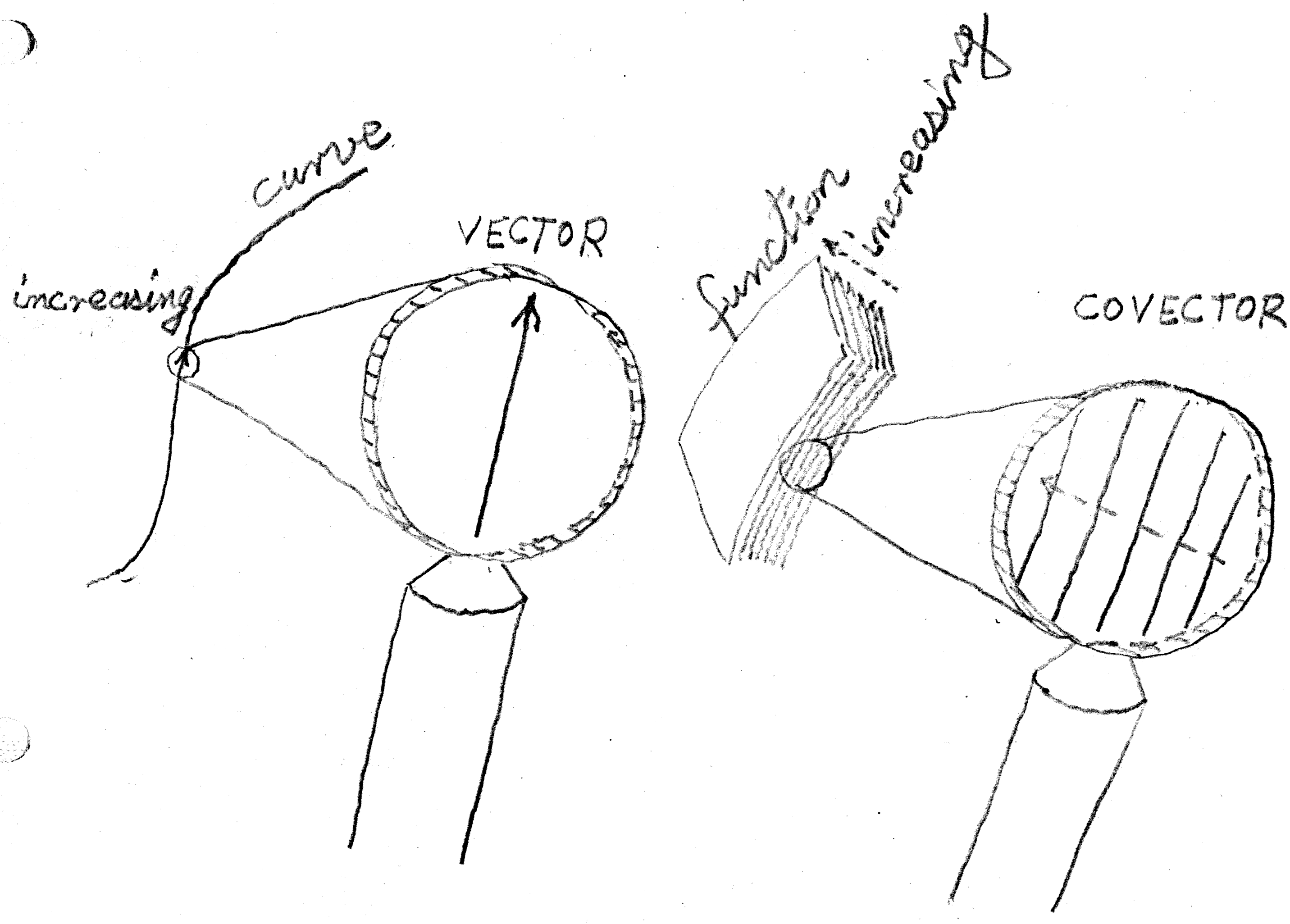
LECTURE 7

Linear functions on a vectorspace V

The Dual Vectorspace V^*

Dirac's Bracket Notation

Basis vs Dual Basis



curve
increasing
VECTOR
function
increasing
COVECTOR

Algebra and Geometry of Dual Vector Spaces

I. a) The space of duals is a space of linear functions
linear functionals
covectors
duals

A **dual** is a derived concept, it depends on the existence and knowledge of a vector space. More precisely, we have the following

Definition 7 (Linear function)

Let V be a vector space.
Consider a scalar valued linear function f defined on V as follows:

$$f: V \longrightarrow \text{scalars}$$
$$x \rightsquigarrow f(x)$$

such that

$$\boxed{f(\alpha x+\beta y) = \alpha f(x) + \beta f(y)} \qquad x, y \in V, \quad \alpha, \beta \in \{\text{scalars}\}$$

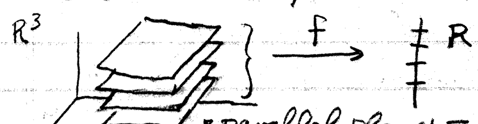
(This is different from non-linear f'ns, obviously!)

Example 1: Let $V = R^n = \{(x_1, x_2, \ldots, x_n)^T\}$

Then $f: R^n \longrightarrow R$

Then $(x_1, x_2, \ldots, x_n)^T \rightsquigarrow f(x_1, \ldots, x_n) = \xi_1 x_1 + \cdots + \xi_n x_n$

is a linear function on R^n. Note that f is determined totally by $\xi_1, \ldots, \xi_n$.



parallel planes = loci of pts where f = constant

Example 2

Consider $V = C[a,b]$, the vector space of functions ψ continuous on $[a,b]$:

$$V = C[a,b] = \{\psi : \psi(s) \text{ is continuous on } [a,b]\}$$

We shall now consider three different linear scalar-valued functions on V:

(i) For any point $s_1 \in [a,b]$ the "s_1-evaluation map" (i.e. the "s_1-sampling function") f

$$V = C(a,b) \xrightarrow{\ f\ } R \qquad \text{(reals)}$$
$$\psi \rightsquigarrow f(\psi) = \psi(s_1)$$

f is a linear function because

$$f(c_1\psi + c_2\phi) = c_1 \psi(s_1) + c_2 \phi(s_1)$$
$$= c_1 f(\psi) + c_2 f(\phi)$$

(ii) The function g defined by

$$V \xrightarrow{g} R$$

$$\psi \rightsquigarrow g(\psi) = \sum_{j=1}^{n} k_j \, \psi(s_j),$$

where $\{s_j\}_1^n$ is some specified collection of points in $[a, b]$, and $\{k_j\}_1^n$ is a set of scalars,

is also a linear function on V.

(iii) Similarly, the map h defined defined by

$$h(\psi) = \int_a^b \psi(s)\, ds$$

is also a linear function on $V = C[a,b]$

Example 3 (skip this one in class)

Consider the vector space

$$V = C^\infty(a,b) = \{\psi : \psi \text{ is } C^\infty \text{ on } (a,b)\}$$

of functions infinitely differentiable on (a,b). Furthermore, let

$$d^j\psi(s) = \left.\frac{d^j\psi}{dx^j}\right|_{x=s}$$

be the j^{th} derivative of ψ at $x=s$.

Then for any fixed $s \in (a,b)$, the map h defined by

$$V = C^\infty(a,b) \xrightarrow{h} R$$

$$\psi \rightsquigarrow h(\psi) = \sum_{j=1}^{n} a_j \, d^j\psi(s)$$

is a linear function on $V = C^\infty(a,b)$

II. The Vector Space V^* dual to V.

Given a vector space V, the consideration of all possible linear functions defined on V gives rise to

$$V^* = \text{set of all linear functions on } V.$$

These linear functions form a vector space in its own right, the dual space of V. Indeed, one has the following

> **Theorem 8**
>
> The set V^* is a vector space

Comment. a) The elements of the vector space V^* of duals are called <u>covectors</u>.

b) That V^* does indeed form a vector space is verified by observing that the collection of linear functions satisfies the 10 properties of a vector space defined on page 1.7

If f, g, h are linear functions (i.e. elements of V^*, and $\alpha, \beta \in R$, then

1. $f+g$ is also a <u>linear</u> function defined by

$$(f+g)(\vec{x}) = f(\vec{x}) + g(\vec{x}) \qquad \forall \vec{x} \in V$$

such that

a) $f+g = g+f$
b) $f+(g+h) = (f+g)+h$
c) zero element is the constant zero function
d) the additive inverse of f is $-f$

2. αf is a <u>linear</u> function defined by

$$(\alpha f)(\vec{x}) = \alpha f(\vec{x}) \qquad \forall \vec{x} \in V$$

such that 7.8

a) $\alpha(\beta f) = (\alpha\beta) f$

b) $1 \cdot f = f$

c) $\alpha(f+g) = \alpha f + \alpha g$

d) $(\alpha+\beta) f = \alpha f + \beta f$

III Dirac's Bracket Notation

To emphasize the duality between the two vector spaces one introduces the bra-cket notation, which is familiar from quantum mechanics

If f is a linear function on V and $f(x)$ it its value at $x \in V$, then one also writes

$$f(x) = \langle f | x \rangle = \langle \underset{\sim}{f} | \vec{x} \rangle . \qquad (\star)$$

Thus the twiddle under f is a reminder that $\underset{\sim}{f} \in V^*$, while x, or better $\vec{x}$, is an element of V.

We say that f operates on the vector x and produces

$$\langle f | x \rangle \qquad (\in R)$$

To emphasize that f is a linear "machine", we write

$$f = \langle \underset{\sim}{f} | \qquad (\in V^*)$$

for the covector, and

$$\vec{x} = | \vec{x} \rangle \qquad (\in V)$$

for the vector. They combine to form

$$\langle \underset{\sim}{f} | \vec{x} \rangle \quad \in R$$

IV Construction of a Linear Function.

The existence and uniqueness of a vector's scalar coefficients relative to a chosen basis is what makes the concept of duality so important geometrically and computationally.

This assessment arises from the following:

Duality Principle.

For each chosen basis of a finite dimensional vector space V, there exists a corresponding basis for V^* and vice versa.

The validation of this principle consists of the actual 3-step construction of the basis dual to the given basis, which we denote by

$B = \{e_1, e_2, \dots, e_n\} \subset V$ (basis for V)

Step 1.

For all vectors, e.g. x and y, one has the unique expansions

$$x = \alpha^1 e_1 + \dots + \alpha^n e_n \qquad (*)$$

$$y = \beta^1 e_1 + \dots + \beta^n e_n$$

$$x+y = (\alpha^1+\beta^1) e_1 + \dots + (\alpha^n+\beta^n) e_n$$

$$cx = c\alpha^1 e_1 + \dots + c\alpha^n e_n$$

Note that the scalar

α^1 is uniquely determined by x

β^1 " " " " y

$\alpha^1+\beta^1$ " " " " $x+y$

$c\alpha^1$ " " " " cx

Step II

These four relation determine a linear function, call it $\underset{\sim}{\omega}^1$. Its defining properties are

$$\left.\begin{aligned} \omega^1(x) &= \alpha^1 \\ \omega^1(y) &= \beta^1 \\ \omega^1(x+y) &= \alpha^1+\beta^1 \\ \omega^1(cx) &= c\,\alpha^1 \end{aligned}\right\}$$

which imply

$$\omega^1(x+y) = \omega^1(x) + \omega^1(y)$$
$$\omega^1(cx) = c\,\omega^1(x)\,.$$

In particular, using Eq.(*) on page 7.12, one has

$$\omega^1(e_1) = 1$$
$$\omega^1(e_2) = 0$$
$$\vdots$$
$$\omega^1(e_n) = 0$$

We conclude that $\underset{\sim}{\omega}^1$ is linear indeed.

$\boxed{\omega^1 \text{ is called the first coordinate function.}}$

Step III

Similarly one defines $\underset{\sim}{\omega}^j$, the j^{th} coordinate function, by

$$\underset{\sim}{\omega}^j(x) = \alpha^j \quad \text{for } j = 2, 3, \cdots, n$$

By applying $\underset{\sim}{\omega}^j$ to the i^{th} basis vector e_i, i.e. by using Eq.(*) on page 7.13 and Eq(*) on page 7.9 we have

$$\underset{\sim}{\omega}^j(\vec{e}_i) \equiv \langle \underset{\sim}{\omega}^j | e_i \rangle = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

or in terms of the Kronecker delta

$$\boxed{\langle \underset{\sim}{\omega}^j | \vec{e}_i \rangle = \delta^j_i}\,.$$

This is called a duality relation. The choice of another basis would have resulted in a different set of coordinate functions, To be done in Lecture 8

but would have again resulted in a duality relation. The corresponding coordinate functions constitute a basis for the dual space V^*. This fact is expressed by the following

Theorem 9 ("Dual basis")

Given: a basis $B = \{e_1, \dots, e_n\}$ for V

Conclusion: The set of linear functions $\{\omega^j\}$ which satisfies the duality relation

$$\langle \omega^j | e_i \rangle = \delta^j_i$$

is a basis for V^*.

Comment: as a reminder we have

ω^j = jth coord. function on any vector

e_i = ith basis vector.

proof:

Let $\underset{\sim}{f} \in V^*$ be some linear function on V

To evaluate $f(\vec{x})$, note that $\vec{x} = \sum \alpha^i \vec{e}_i$.

Thus

$$\begin{aligned} f(\vec{x}) &= f(\textstyle\sum_i \alpha^i \vec{e}_i) \\ &= \sum_i f(\vec{e}_i)\,\alpha^i \qquad \alpha^i = i^{th} \text{ coord of } x \\ &= \sum f(e_i)\,\omega^i(\vec{x}) \qquad \forall \vec{x} \end{aligned}$$

Hence

$$\boxed{f = \sum_i f(e_i)\,\underset{\sim}{\omega}^i}$$

which is the expansion of $\underset{\sim}{f}$ w.r.t. the basis B.

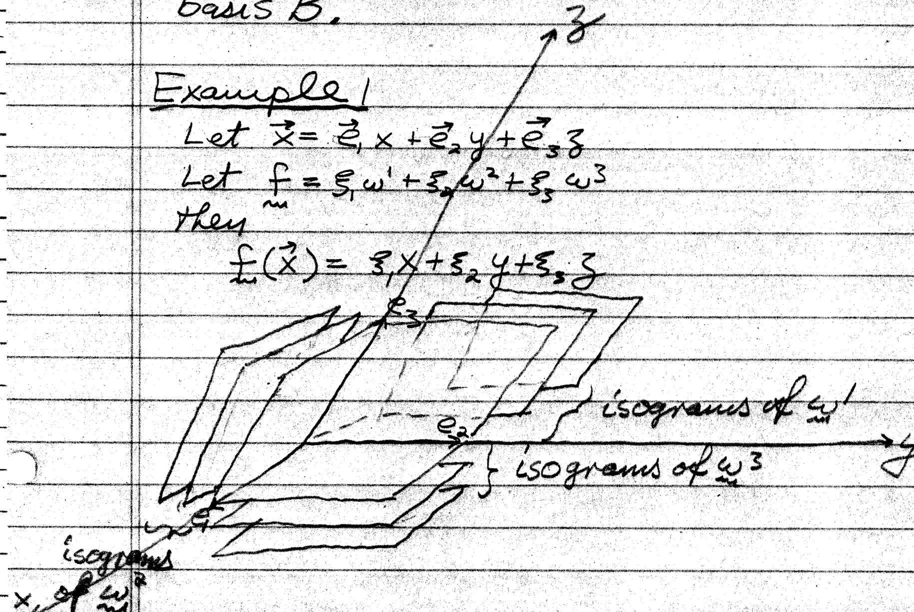
Example 1

Let $\vec{x} = \vec{e}_1 x + \vec{e}_2 y + \vec{e}_3 z$

Let $\underset{\sim}{f} = \xi_1 \omega^1 + \xi_2 \omega^2 + \xi_3 \omega^3$

then

$$\underset{\sim}{f}(\vec{x}) = \xi_1 x + \xi_2 y + \xi_3 z$$



To be done in Lecture 8

Example 2 (Columnspace* = Row space)

GIVEN:

Let $B = \{ e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}; e_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}; e_3 = \begin{bmatrix}1\\1\\1\end{bmatrix} \}$ be a basis for the column space $V = R^3$

a) Identify V^*, the space dual V.

b) Find the basis $B^* = \{ \underset{\sim}{\omega}^1, \underset{\sim}{\omega}^2, \underset{\sim}{\omega}^3 \}$ dual to B, i.e. $\langle \underset{\sim}{\omega}^j | e_i \rangle = \delta^j_i$.

Solution:

a) The dual space V^* consists of the row space $V^* = \{ \underset{\sim}{\sigma} = [a\ b\ c] : a, b, c \in R^1 \} (= R^3)$

Indeed, for any $\vec{x} = \begin{bmatrix}x\\y\\z\end{bmatrix} \in R^3$

$$\underset{\sim}{\sigma}(\vec{x}) = \langle \underset{\sim}{\sigma} | \vec{x} \rangle = [a\ b\ c]\begin{bmatrix}x\\y\\z\end{bmatrix} = ax + by + cz$$

$$\equiv \sigma(x, y, z)$$

b) Each ω^j is a row vector. They

must satisfy

$$\left.\begin{aligned} \langle \omega^1 | e_1 \rangle &= [a\ b\ c]\begin{bmatrix}1\\0\\0\end{bmatrix} = 1 \\ \langle \omega^1 | e_2 \rangle &= [a\ b\ c]\begin{bmatrix}1\\1\\0\end{bmatrix} = 0 \\ \langle \omega^1 | e_3 \rangle &= [a\ b\ c]\begin{bmatrix}1\\1\\1\end{bmatrix} = 0 \end{aligned}\right\} \Rightarrow \omega^1 = [1\ {-1}\ 0]$$

$$\left.\begin{aligned} \langle \omega^2 | e_1 \rangle &= [d\ e\ f]\begin{bmatrix}1\\0\\0\end{bmatrix} = 0 \\ \langle \omega^2 | e_2 \rangle &= [d\ e\ f]\begin{bmatrix}1\\1\\0\end{bmatrix} = 1 \\ \langle \omega^3 | e_3 \rangle &= [d\ e\ f]\begin{bmatrix}1\\1\\1\end{bmatrix} = 0 \end{aligned}\right\} \Rightarrow \omega^2 = [0\ 1\ {-1}]$$

$$\left.\begin{aligned} \langle \omega^3 | e_1 \rangle &= [u\ v\ w]\begin{bmatrix}1\\0\\0\end{bmatrix} = 0 \\ \langle \omega^3 | e_2 \rangle &= [u\ v\ w]\begin{bmatrix}1\\1\\0\end{bmatrix} = 0 \\ \langle \omega^3 | e_3 \rangle &= [u\ v\ w]\begin{bmatrix}1\\1\\1\end{bmatrix} = 1 \end{aligned}\right\} \underset{\sim}{\omega}^3 = [0\ 0\ 1]$$

Thus

$$B^* = \{\omega^j\}_1^3 = \{ [1\ {-1}\ 0], [0\ 1\ {-1}], [0\ 0\ 1] \}$$

Example 3

If $B = \{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix} \}$ By contrast with Example 2, then

$$B^* = \{ [1\ {-1}\ 0], [0\ 1\ {-\tfrac{1}{2}}], [0, 0, \tfrac{1}{2}] \}$$

To be done in 7.19 Lecture

Note that changing only one element of B changes <u>two</u> elements of B^*. This implies that there is as-yet no basis independent correspondence between V^* and V. Such a correspondence would have required that changing only one of the basis vectors in V would have produced a corresponding change in only one basis vector in V^*.