LECTURE 8

1. The Duality Principle

2. Example for $\mathbb{R}^n$: $\{\text{Column vectors}\} = V$
$\{\text{Row vectors}\} = V^* =$ dual space

3. Addition of vectors and covectors

Proposition (The Duality Principle)

For each chosen basis

$$\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$$

of a finite dimensional vector space V, there exists a corresponding basis

$$\{\underset{\sim}{\omega}^1, \underset{\sim}{\omega}^2, \ldots, \underset{\sim}{\omega}^n\}$$

for V^*, and vice versa, such that

$$\langle \underset{\sim}{\omega}^j | \vec{e}_i \rangle = \delta^j{}_i$$

Comment

The evaluation

$$\langle \underset{\sim}{\omega}^j | \vec{e}_i \rangle \equiv \underset{\sim}{\omega}^j(\vec{e}_i) = \delta^j{}_i$$

is not to be confused with an "inner product," which will be introduced later and which conceptualizes the idea of length and angle. Here we leave these quantities unspecified.

The validation of the duality principle consists of the actual (3-step) construction of the basis dual to the given basis, which we denote by

$$B = \{e_1, e_2, \ldots, e_n\} \subset V \qquad \text{(basis for } V\text{)}$$

Step 1.

For all vectors, e.g. x and y one has the unique expansions

$$x = \alpha^1 e_1 + \cdots + \alpha^n e_n \qquad (*)$$

$$y = \beta^1 e_1 + \cdots + \beta^n e_n$$

$$x+y = (\alpha^1+\beta^1) e_1 + \cdots + (\alpha^n+\beta^n) e_n$$

$$cx = c\alpha^1 e_1 + \cdots + c\alpha^n e_n$$

Note that the scalar

α^1	is	uniquely	determined	by	x
β^1	"	"	"	"	y
$\alpha^1+\beta^1$	"	"	"	"	$x+y$
$c\alpha^1$	"	"	"	"	cx

Step II
These four relation determine a linear function, call it $\underset{\sim}{\omega}^1$. Its defining properties are

$$\left.\begin{aligned} \omega^1(x) &= \alpha^1 \\ \omega^1(y) &= \beta^1 \\ \omega^1(x+y) &= \alpha^1+\beta^1 \\ \omega^1(cx) &= c\,\alpha^1 \end{aligned}\right\}$$

which imply

$$\omega^1(x+y) = \omega^1(x) + \omega^1(y)$$
$$\omega^1(cx) = c\,\omega^1(x)$$

In particular, using Eq. (*) on page 7.12, one has

$$\omega^1(e_1) = 1$$
$$\omega^1(e_2) = 0$$
$$\vdots$$
$$\omega^1(e_n) = 0$$

We conclude that $\underset{\sim}{\omega}^1$ is linear indeed.

ω^1 is called the first coordinate function.

Step III
Similarly one defines $\underset{\sim}{\omega}^j$, the j^{th} coordinate function, by

$$\underset{\sim}{\omega}^j(x) = \alpha^j \quad \text{for } j = 2, 3, \ldots, n$$

By applying $\underset{\sim}{\omega}^j$ to the i^{th} basis vector e_i, i.e. by using Eq. (*) on page 7.13 and Eq. (*) on page 7.9 we have

$$\underset{\sim}{\omega}^j(\vec{e}_i) \equiv \langle \underset{\sim}{\omega}^j | e_i \rangle = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

or in terms of the Kronecker delta

$$\boxed{\langle \underset{\sim}{\omega}^j | \vec{e}_i \rangle = \delta^j_i}.$$

This is called a duality relation or duality principle.
The choice of another basis would have resulted in a different set of coordinate functions.

but would have again resulted in a duality relation. The corresponding coordinate functions constitute a basis for the dual space V^*. This fact is expressed by the following

Theorem 9 ("Dual basis")

Given: a basis $B=\{e_1, \dots, e_n\}$ for V

Conclusion: The set of linear functions $\{\underset{\sim}{\omega}^j\}$ which satisfies the duality relation

$$\langle \underset{\sim}{\omega}^j | \vec{e}_i \rangle = \delta^j{}_i$$

is a basis for V^*.

Comment: as a reminder we have

$\underset{\sim}{\omega}^j$ = jth coord. function on any vector

$\vec{e}_i$ = ith basis vector

proof:

Let $\underset{\sim}{f} \in V^*$ be some linear function on V

To evaluate $f(\vec{x})$, note that $\vec{x}=\sum \alpha^i \vec{e}_i$.

Thus

$$f(\vec{x}) = f(\textstyle\sum_i \alpha^i \vec{e}_i)$$
$$= \sum_i f(\vec{e}_i)\, \alpha^i \qquad \alpha^i = i^{th} \text{ coord of } x$$
$$= \sum f(e_i)\, \omega^i(\vec{x}) \qquad \forall \vec{x}$$

Hence

$$\boxed{f = \sum_i f(e_i)\, \underset{\sim}{\omega}^i}$$

which is the expansion of $\underset{\sim}{f}$ w.r.t. the basis B.

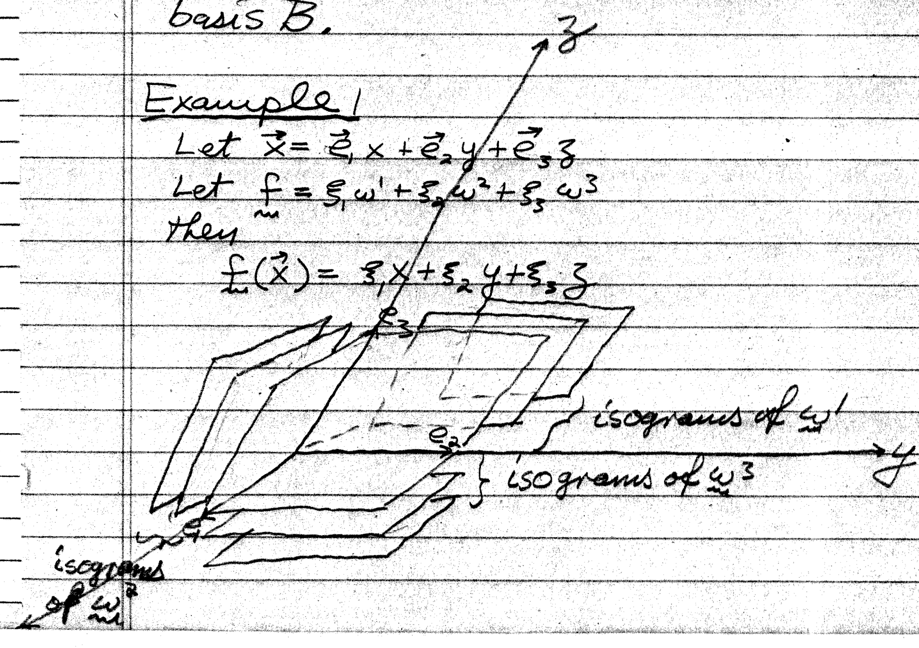
Example 1

Let $\vec{x} = \vec{e}_1 x + \vec{e}_2 y + \vec{e}_3 z$

Let $\underset{\sim}{f} = \xi_1 \omega^1 + \xi_2 \omega^2 + \xi_3 \omega^3$

then

$$\underset{\sim}{f}(\vec{x}) = \xi_1 x + \xi_2 y + \xi_3 z$$



Example 2 (Columnspace* = Rowspace)

GIVEN:

Let $B = \{ e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}; e_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}; e_3 = \begin{bmatrix}1\\1\\1\end{bmatrix} \}$ be a basis for the column space $V = R^3$

a) Identify V^*, the space dual V.

b) Find the basis $B^* = \{ \underset{\sim}{\omega}^1, \underset{\sim}{\omega}^2, \underset{\sim}{\omega}^3 \}$ dual to B, i.e. $\langle \underset{\sim}{\omega}^j | e_i \rangle = \delta^j_i$

Solution:

a) The dual space V^* consists of the row space $V^* = \{ \underset{\sim}{\sigma} = [a\,b\,c] : a, b, c \in R^1 \} (= R^3)$

Indeed, for any $\vec{x} = \begin{bmatrix}x\\y\\z\end{bmatrix} \in R^3$

$$\langle \underset{\sim}{\sigma} | \vec{x} \rangle \equiv \underset{\sim}{\sigma}\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = [a\,b\,c]\begin{bmatrix}x\\y\\z\end{bmatrix} = ax + by + cz$$

$$\equiv \sigma(x, y, z)$$

b) Each $\underset{\sim}{\omega}^j$ is a row vector. They

must satisfy

$$\left.\begin{aligned} \langle \underset{\sim}{\omega}^1 | \vec{e}_1 \rangle &= [a\,b\,c]\begin{bmatrix}1\\0\\0\end{bmatrix} = 1 \\ \langle \underset{\sim}{\omega}^1 | \vec{e}_2 \rangle &= [a\,b\,c]\begin{bmatrix}1\\1\\0\end{bmatrix} = 0 \\ \langle \underset{\sim}{\omega}^1 | e_3 \rangle &= [a\,b\,c]\begin{bmatrix}1\\1\\1\end{bmatrix} = 0 \end{aligned}\right\} \Rightarrow \underset{\sim}{\omega}^1 = [1\,-1\,0]$$

$$\left.\begin{aligned} \langle \underset{\sim}{\omega}^2 | \vec{e}_1 \rangle &= [d\,e\,f]\begin{bmatrix}1\\0\\0\end{bmatrix} = 0 \\ \langle \underset{\sim}{\omega}^2 | \vec{e}_2 \rangle &= [d\,e\,f]\begin{bmatrix}1\\1\\0\end{bmatrix} = 1 \\ \langle \underset{\sim}{\omega}^3 | \vec{e}_3 \rangle &= [d\,e\,f]\begin{bmatrix}1\\1\\1\end{bmatrix} = 0 \end{aligned}\right\} \Rightarrow \omega^2 = [0\,1\,-1]$$

$$\left.\begin{aligned} \langle \underset{\sim}{\omega}^3 | \vec{e}_1 \rangle &= [u\,v\,w]\begin{bmatrix}1\\0\\0\end{bmatrix} = 0 \\ \langle \underset{\sim}{\omega}^3 | \vec{e}_2 \rangle &= [u\,v\,w]\begin{bmatrix}1\\1\\0\end{bmatrix} = 0 \\ \langle \underset{\sim}{\omega}^3 | \vec{e}_3 \rangle &= [u\,v\,w]\begin{bmatrix}1\\1\\1\end{bmatrix} = 1 \end{aligned}\right\} \underset{\sim}{\omega}^3 = [0\,0\,1]$$

Thus

$$B^* = \{ \omega^j \}_1^3 = \{ [1\,-1\,0], [0\,1\,-1], [0\,0\,1] \}$$

Example 3

If $B = \{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix} \}$

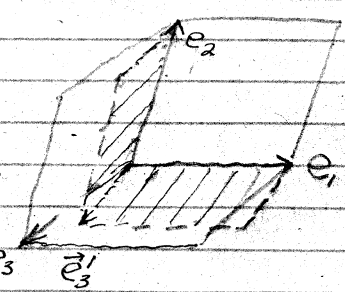
then

$$B^* = \{ [1\,-1\,0], [0\,1\,-\tfrac{1}{2}], [0, 0, \tfrac{1}{2}] \}$$

Comment

Note that changing only one element of B changes two elements of B^*. This implies that there is as-yet no basis independent correspondence between V^* and V. Such a correspondence would have required that changing only one of the basis vectors in V would have produced a corresponding change in only one basis vector in V^*. Geometrically we have

$\{\vec{e}_1\ \vec{e}_2\ \vec{e}_3\}$ vs $\{\vec{e}_1, \vec{e}_2, \vec{e}\,'_3\}$



$\{\underset{\sim}{\omega}^1, \underset{\sim}{\omega}^2\ \underset{\sim}{\omega}^3\}$ vs $\{\underset{\sim}{\omega}'^1, \underset{\sim}{\omega}'^2, \underset{\sim}{\omega}'^3\}$

where $e_3 \neq e'_3$

where $\omega^2 \neq \omega'^2$, $\omega^3 \neq \omega'^3$

Summary of comment on pages 8.5-8.9

The exists a unique correspondence between ordered basis sets in V and V^*

$$\{\vec{e}_i\} \longleftrightarrow \{\underset{\sim}{\omega}^j\}$$

but not between individual vectors in V and V^*:

$$\{\text{coord. vects}\} \not\longleftrightarrow \{\text{coord. surfaces}\}$$

More succinctly one says that there exist no natural (i.e basis independent) isomorphim between V and V^*;

However we shall see in Lecture 9 that if V is endowed with an inner product, then there is a natural isomorphism between V and V^*.

VII

Geometry of duals in an oblique coordinate system.

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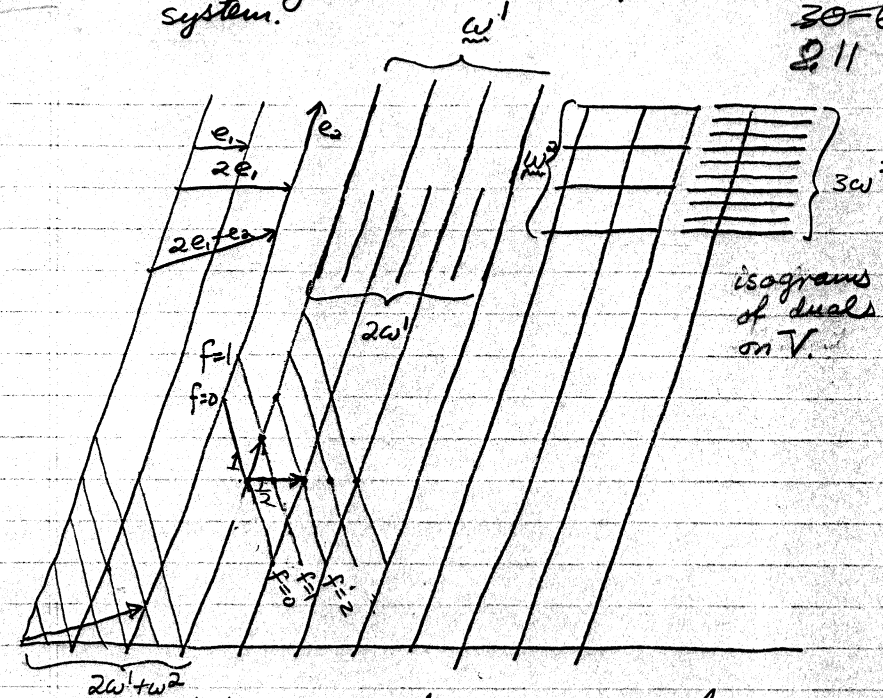

Addition of vectors and covectors

$\underset{\sim}{f} = 2\omega^1+\omega^2$ is determined from $\langle \underset{\sim}{f} \mid e_1\rangle = 2$ (bongs

$\langle \underset{\sim}{f} \mid e_2\rangle = 1$ (bong:

The density of the blades of $\underset{\sim}{f}$ into the direction e_1 is 2, i.e. the spacing between successive integral valued level surfaces is $\frac{1}{2}$

$\langle 2\omega^1+\omega^2 \mid 2e_1+e_2\rangle = 4+0+0+1=5$ (bongs!)