LECTURE 8 (APPENDIX)

Example 0: Dollar values of fruit inventories

Interpolation of sampled data expressed in terms of sampling functions

Examples: Lagrangian interpolation via quadratic polynomials

Example 2: Roof functions

Example 3: Band-limited basis functions
Example 1 (Dollar values of fruit inventories)

The concept of a vector space \((V)\)
and its space of covectors \((V^*)\)
makes up the dual vector space \((V^+)\).

are so ubiquitous that it takes no scientific specialized knowledge or training to grasp and use these concepts.

The ensuing example illustrates this.

Example (Linear function defined on the space of fruit inventories)

Consider a fruit inventory consisting of \(a\) apples, \(b\) bananas, \(c\) coconuts.

We designate

\[
\begin{align*}
1\text{ apple} &= e_a^* \\
1\text{ banana} &= e_b^* \\
1\text{ coconut} &= e_c^*
\end{align*}
\]

The fruit inventory is therefore

\[
x = \alpha^a e_a^* + \alpha^b e_b^* + \alpha^c e_c^* + \ldots
\]

Here

\[
\begin{align*}
\alpha^a &= \text{quantity of apples} \\
\alpha^b &= \text{quantity of bananas} \\
\alpha^c &= \text{quantity of coconuts}
\end{align*}
\]

A) Consider a physical system which (i) recognizes apples to the exclusion of all other fruit, and (ii) measures the quantity of apples \(\alpha^a\) for any given fruit inventory \(x\).
Given some fruit inventory, operating on $y$ yields $\beta_0$, the quantity of apples.

By the same token, an analogous "banana recognition/measuring system" operating on $y$ yields $\beta_1$, the quantity of bananas, and a "coconut recognition/measuring system" yields $\beta_2$, the quantity of coconuts.

Thus we have three linear systems, $\omega^a$, $\omega^b$, and $\omega^c$:

\[
\begin{align*}
\omega^a &: \omega^a(y) = \alpha^a \\
\omega^b &: \omega^b(y) = \beta^b \\
\omega^c &: \omega^c(y) = \beta^c \\
\end{align*}
\]

Their recognition and measuring processes obey the following linear mathematical laws:

\[
\begin{align*}
\omega^a &: \omega^a(x) = \alpha^a \\
\omega^a(y) = \beta^a \\
\omega^a(x+y) = \alpha^a + \beta^a = \omega^a(x) + \omega^a(y) \\
\omega^a(cx) = c\alpha^a = c\omega^a(x) \\
\end{align*}
\]

\[
\begin{align*}
\omega^b &: \omega^b(x) = \alpha^b \\
\omega^b(y) = \beta^b \\
\omega^b(x+y) = \alpha^b + \beta^b = \omega^b(x) + \omega^b(y) \\
\omega^b(cx) = c\alpha^b = c\omega^b(x) \\
\end{align*}
\]

\[
\begin{align*}
\omega^c &: \omega^c(x) = \alpha^c \\
\omega^c(y) = \beta^c \\
\omega^c(x+y) = \alpha^c + \beta^c = \omega^c(x) + \omega^c(y) \\
\omega^c(cx) = c\alpha^c = c\omega^c(x) \\
\end{align*}
\]
Suppose that the Federal Reserve prints extra dollars and injects them into general circulation. With more dollars chasing the same fruit inventories, let us say that the dollar of each fruit item gets bid up by \$\delta_a, \$\delta_b, \$\delta_c.$

In other words, the dollar pricing in terms of the diluted dollars is increased by

\[ \$ = \$a \alpha_a^2 + \$b \alpha_b^4 + \$c \alpha_c^6 + \$d \alpha_d^8 + \$e \alpha_e^{10}. \]

and the new pricing scheme is

\[ \$ + \$ = (\$a + \$d) \alpha_a^2 (\$b + \$c) \alpha_b^4 (\$e + \$f) \alpha_e^{10}. \]

The value of the fruit inventory is

\[ \$ \]
in units of inflated dollars is

\[ \langle \$ + \$ \rangle^x = \langle \$ + \$ \rangle^x \] 

\[ = (x^a + (x^a + x^b + x^c))^x \] 

\[ = (x^a + x^b + x^c) = \langle \$ \rangle^x \] 

The price increase,

\[ \Delta \alpha^a + \Delta \alpha^b \alpha^x + \Delta \alpha^c \alpha^x = \langle \Delta \$ \rangle^x \] 

is not an increase in its value, but is the result of inflated dollars.
Suppose we wish to do many experiments, and each experiment consists of a set of measurements. Each set is expressed by, say, \( n+1 \) data points\( (x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n) \) for each experiment. We stipulate that all experiments have the same fixed set of sampled points \( \{x_0, x_1, \ldots, x_n\} \) and that there be one set of measured values \( y_0, y_1, \ldots, y_n \) for each experiment.

The problem is to fit a curve, say \( \psi(x) \), of a certain type so that
\[
\psi(x_i) = y_i, \quad i = 0, 1, \ldots, n
\]
for this experiment.

More precisely, one has the following problem:

**GIVEN:**

(i) An \( (n+1) \)-dimensional subspace \( V = C[a, b] \) of functions \( \psi \) continuous on \([a, b]\), and

(ii) \( a_0 < a_1 < \cdots < a_n \in [a, b] \).

For each experiment \( \{(x_0, y_0), \ldots, (x_n, y_n)\} \)

**FIND:**

That function \( \psi \in V \) which has the property
\[
(\psi(x_i), y_i) = \langle \psi, y_i \rangle, \quad i = 0, 1, \ldots, n
\]
i.e. the graph of \( \psi \) is supposed to pass through the data points.

Eq(4) on page 89.1

\[(x_0, y_0) \cdots (x_n, y_n)\]

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**SOLUTION for** \( V = \mathbb{P}_2 = \{ \psi : \psi(x) = a_0 + a_1 x + a_2 x^2, a_0, a_1, a_2 \in \mathbb{R} \} \)

1. Consider the following evaluation ("sampling") maps \( \omega^0, \omega^1, \omega^2 \in V^* \): \(
\langle \omega^0 \psi \rangle = \omega^0(\psi) = \psi(x_0)
\langle \omega^1 \psi \rangle = \omega^1(\psi) = \psi(x_1)
\langle \omega^2 \psi \rangle = \omega^2(\psi) = \psi(x_2)
\)

Note: The notation \( \langle \cdot, \cdot \rangle \) denotes the evaluation at the points \( x_0, x_1, x_2 \).
Example 1 (only 3 sample points)

Result of 1st sampling experiment:
\[ \{ y_0^{(1)}, y_1^{(1)}, y_2^{(1)} \} \]
\[ \psi^{(1)} = 22 \]

Result of kth sampling experiment:
\[ \{ y_0^{(k)}, y_1^{(k)}, y_2^{(k)} \} \]
\[ \psi^{(k)} = ?? \]

Problem:
For any sampling run whose results is
\[ \{ y_0, y_1, y_2 \} \]
FIND \( y = a_0 + a_1 x + a_2 x^2 \in \mathbb{R}_2 \)
such that
\[ \psi(a_0) = y_0 \]
\[ \psi(a_1) = y_1 \]
\[ \psi(a_2) = y_2 \]

Solution (in 3 steps)

Step 1. Consider the following three evaluation ("sampling") maps:
\[ \omega_0, \omega_1, \omega_2 \in V^* \text{ (the space dual to } \mathbb{R}_2 \text{) } \]
\[ \langle \omega_0 | \psi \rangle = \omega_0(\psi) = \psi(a_0) \]
\[ \langle \omega_1 | \psi \rangle = \omega_1(\psi) = \psi(a_1) \]
\[ \langle \omega_2 | \psi \rangle = \omega_2(\psi) = \psi(a_2) \]

Dirac Notation

All sampling experiments are performed at the same x-values namely,
\[ \{ x = x_0, x = x_1, x = x_2 \} \]
The result of the kth sampling run is
\[ \{ y_0^{(k)}, y_1^{(k)}, y_2^{(k)} \} \]
They constitute a linearly independent set (basis) for \( V^* \)

\[ B^* = \{ \omega^0, \omega^1, \omega^2 \} \subset V^* = \mathbb{R}^2 \]

step 2 The solution to this example problem consists of finding the basis

\[ B = \{ \psi_0, \psi_1, \psi_2 \} \subset \mathbb{R}^2 \text{ for } V = \mathbb{R}^2 \]

which satisfies the duality principle ("duality relation")

\[ \langle \psi_i, \psi_j \rangle = \omega^i(\psi_j) = \psi_j(\omega^i) = \delta^i_j. \]

Comment: Notice that here in step 2 we are asked to execute the converse of Theorem 9 on page 3.5, namely, namely given

\[ B^* = \{ \omega^0, \omega^1, \omega^2 \} \text{ for } V^* \]

construct vectors

\[ \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2 \equiv B \]

such that

\[ \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta^i_j. \]

They constitute a lin. indep. spanning set for \( V \), i.e., they form a basis

\[ B^* = \{ \omega^0, \omega^1, \omega^2 \} \text{ for } V^* = \mathbb{R}^2 \]

2. Let \( B = \{ \psi_0, \psi_1, \psi_2 \} \subset \mathbb{R}^2 \) be the basis.

Such a basis satisfies

\[ \langle \omega^i, \psi_j \rangle = \omega^i(\psi_j) = \psi_j(\omega^i) = \delta^i_j. \]

The explicit formulae for these basis elements have been exhibited by the Enlightenment mathematician Lagrange:

\[ \psi_0(\omega) = \frac{(\omega_0 - \omega_0)(\omega_1 - \omega_2)}{(a_1 - a_0)(a_2 - a_0)} \]

\[ \psi_1(\omega) = \frac{(\omega_0 - \omega_0)(\omega_0 - \omega_2)}{(a_1 - a_0)(a_2 - a_0)} \]

\[ \psi_2(\omega) = \frac{(\omega_0 - \omega_0)(\omega_0 - \omega_1)}{(a_1 - a_0)(a_2 - a_0)} \]

They satisfy the required duality relation

\[ \psi_j(\omega_i) = \delta^i_j = \{ 1 \text{ if } i = j, 0 \text{ otherwise} \} \]
The sought-after function $\Psi$

$$\Psi(x) = \Psi_0(x) + \Psi_1(x) + \Psi_2(x)$$

**Step 3:**

The function $\Psi$ is a quadratic whose graph passes through the data points Eq. (4) near the top of page 154.

It is the linear combination of three simple quadratics:

$$\Psi = \Psi_0 + \Psi_1 + \Psi_2$$

And as mentioned on the previous page, these three quadratic are basis vectors for $V = \mathbb{R}_2$, dual to the three basis covectors $\omega^0, \omega^1, \omega^2$ for $V^*$.

Claim: $V$ is a vector space.

Consider the basis $\{\Psi_i : i = 0, \ldots, n\} \subseteq V$ with the property

$$\Psi_i(x_j) = \delta_{ij} = \langle \omega^i | \Psi_j \rangle$$

Example 2

$V$ - space of continuous piece-wise linear functions on the closed interval $[x_0, x_n]$

$$= \text{CPL}([x_0, \ldots, x_n])$$

where $x_0 < x_1 < x_2 \ldots < x_n$

and $\Psi \in V$ is linear in each of the subintervals $[x_i, x_{i+1}]$ and continuous at each point $x_j,$ $j = 0, \ldots, n$. 
To find the coordinates of $f$, we have
\[ f(x) = \frac{ax}{b + cx} \]
where $a$, $b$, and $c$ are constants.

The solutions are:
\[ f = \frac{a}{b + cx} \]

In terms of the linear sampling model,
\[ f \equiv \frac{a}{b + cx} \]
for $x \in [0, \infty)$. Let $f$ be a weight function with domain $[0, \infty)$.

\[ f(x) = \int_0^x g(t) \, dt \]

**Example:**
\[ f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 2-x & \text{if } 1 < x \leq 2 \\ 0 & \text{if } x > 2 \end{cases} \]
Example 3: For the case of the vector space $V = B_N$ of band-limited function of period $2\pi$, consider the following set of basis functions

$$\psi(x) = \sum_{m=0}^{N} a_m \cos mx + \sum_{m=0}^{N} b_m \sin mx \quad \text{for} \quad \mathbf{e}_{B_N}$$

The amazing thing about the band-limited functions is that, although they are continuous, each one may be reconstructed with 100% accuracy from a set of sampled values.

$$\psi(x) = \sum_{k=0}^{N} \psi(x_k) \quad \mathcal{F}_k$$

(This is a special case of the Whittaker-Shannon sampling theorem.)

Comment: Each of the basis functions is the response of a low-pass filter to a periodic train of impulses.

They belong to $V$ because

$$\psi_R(x) = \sum_{m=-N}^{N} e^{-im(x-x_k)} = 1 + 2 \sum_{m=1}^{N} \cos mx \quad \text{for} \quad \mathbf{e}_{B_N}$$

These basis functions satisfy

$$\psi_k(x_k) = \delta_k \quad \text{(why?)}$$