LECTURE 8 (APPENDIX)

Example 0: Dollar values of fruit inventories

Interpolation of sampled data expressed in terms of sampling functions

Example 1: Lagrangian interpolation via quadratic polynomials

Example 2: Roof functions

Example 3: Band-limited basis functions

Example (Dollar values of fruit inventories)

The concept of a vector space (V) and its space of covectors, which make up the dual vector space (V^*), are so ubiquitous that it takes no specialized scientific knowledge or training to grasp and use these concepts. The ensuing example illustrates this.

Example (Linear $ function defined on the space of fruit inventories)

Consider a fruit inventory consisting of x^a apples, x^b bananas, x^c coconuts. We designate

$$1\text{ apple} = \vec{e}_a,$$
$$1\text{ banana} = \vec{e}_b,$$
$$1\text{ coconut} = \vec{e}_c$$

The fruit inventory is therefore

$$\vec{x} = \alpha^a \vec{e}_a + \alpha^b \vec{e}_b + \alpha^c \vec{e}_c + \cdots$$

Here

α^a = quantity of apples
α^b = quantity of bananas
α^c = quantity of coconuts
$\vdots$

A) Consider a physical system which (i) recognizes apples to the exclusion of all other fruits, and (ii) measures the quantity of apples α^a for any given fruit inventory $\vec{x}$.

Given some fruit inventory,

$$\vec{y} = \beta^a \vec{e}_a + \beta^b \vec{e}_b + \gamma^c \vec{e}_c,$$

this "apple recognition/measuring" system operating on $\vec{y}$ yields β^a, the quantity of apples.

By the same token an analogous "banana recognition/measuring" system operating on $\vec{y}$ yields β^b, the quantity of bananas, and a "coconut recognition/measuring" system yields β^c, the quantity of bananas.

Thus we have three <u>linear</u> systems,

$\underset{\sim}{\omega}^a$, $\underset{\sim}{\omega}^b$ and ω^c:

$$\vec{y} \rightarrow \boxed{\omega^a} \rightarrow \beta^a;\quad \vec{y} \rightarrow \boxed{\omega^b} \rightarrow \beta^b;\quad \vec{y} \rightarrow \boxed{\omega^c} \rightarrow \beta^c$$

Their recognition and measuring processes obey the following linear mathematical laws:

$$\left.\begin{aligned} \omega^a:\quad & \omega^a(\vec{x}) = \alpha^a \\ & \omega^a(\vec{y}) = \beta^a \\ \omega^a(\vec{x}+\vec{y}) &= \alpha^a + \beta^a = \omega^a(\vec{x}) + \omega^a(\vec{y}) \\ \omega^a(c\vec{x}) &= c\alpha^a = c\,\omega^a(\vec{x}) \end{aligned}\right\}\ \text{quantity of apples}$$

$$\left.\begin{aligned} \omega^b:\quad & \omega^b(\vec{x}) = \alpha^b \\ & \omega^b(y) = \beta^b \\ \omega^b(\vec{x}+\vec{y}) &= \alpha^b + \beta^b = \omega^b(\vec{x}) + \omega^b(\vec{y}) \\ \omega^b(c\vec{x}) &= c\alpha^b = c\,\omega^b(\vec{x}) \end{aligned}\right\}\ \text{quantity of bananas}$$

$$\left.\begin{aligned} \omega^c:\quad & \omega^c(\vec{x}) = \alpha^c \\ & \omega^c(\vec{y}) = \beta^c \\ \omega^c(\vec{x}+\vec{y}) &= \alpha^c + \beta^c = \omega^c(\vec{x}) + \omega^c(\vec{y}) \\ \omega^c(c\vec{x}) &= c\alpha^c = c\,\omega^c(\vec{x}) \end{aligned}\right\}\ \text{quantity of coconuts}$$

B) Next conside the <u>value</u> of an inventory, say $\vec{x}$, in units of dollars

$$\$(\vec{x}) = (\xi_a \underset{\sim}{\omega}^a + \xi_b \underset{\sim}{\omega}^b + \xi_c \underset{\sim}{\omega}^c)(\alpha^a \vec{e}_a + \alpha^b \vec{e}_b + \alpha^c \vec{e}_c)$$

Here

ξ_a = price per apple $\left[\frac{\$}{1\text{ apple}}\right]$

ξ_b = price per banana $\left[\frac{\$}{1\text{ banana}}\right]$

ξ_c = price per coconut $\left[\frac{\$}{\text{coconut}}\right]$

Thus one finds

$$\$(\vec{x}) = \xi_a \alpha^a + \xi_b \alpha^b + \xi_c \alpha^c,$$

which is the value of $\vec{x}$ in terms of dollars.

C) Suppose that the Federal Reserve prints extra dollars and injects them into general circulation. With more dollars chasing the same fruit inventories, let us say that the dollar price of each fruit item gets bid up by

$$\Delta\xi_a, \Delta\xi_b, \Delta\xi_c.$$

In other words, the dollar pricing in terms of the diluted dollars is increased by

$$\Delta\$ = \Delta\xi_a \underset{\sim}{\omega}^a + \Delta\xi_b \underset{\sim}{\omega}^b + \Delta\xi_c \underset{\sim}{\omega}^c$$

and the new pricing scheme is

$$\$ + \Delta\$ = (\xi_a + \Delta\xi_a)\omega^a + (\xi_b + \Delta\xi_b)\omega^b + (\xi_c + \Delta\xi_c)\omega^c;$$

The value of the fruit inventory $\vec{x}$

in units of inflated dollars is

$$\langle \$ + \Delta\$ | \vec{x} \rangle = \langle (\xi_a + \Delta\xi_a)\omega^a + (\xi_b + \Delta\xi_b)\omega^b + (\xi_c + \Delta\xi_c)\omega^c \,|$$

$$| \alpha^a e_a + \alpha^b e_b + \alpha^c e_c \rangle$$

$$= (\xi_a + \Delta\xi_a)\alpha^a + (\xi_b + \Delta\xi_b)\alpha^b + (\xi_c + \Delta\xi_c)\alpha^c$$

The price increase

$$\Delta\xi_a \alpha^a + \Delta\xi_b \alpha^b + \Delta\xi_c \alpha^c = \langle \Delta\$ | \vec{x} \rangle$$

in the sales price of the fruit inventory $\vec{x}$ is not an increase in its value but is the result of inflated dollars.

INTERPOLATION of SAMPLED DATA.

Suppose we wish to do many experiments, and each experiment consists of a set of measurements. Each set is expressed by, say, $n+1$ data points

$$(s_0, y_0), (s_1, y_1), \cdots, (s_n, y_n) \qquad (*)$$

for each experiment.

We stipulate that all experiments have the same fixed set of sampled points

$$\{s_0, s_1, \cdots, s_n\}$$

and that there be one set of measured values

$$y_0, y_1, \cdots, y_n$$

for each experiment.

The problem is to fit a curve, say $\psi(s)$, of a certain type so that

$$\psi(s_j) = y_j \qquad j = 0, 1, \cdots, n$$

for this experiment.

More precisely one has the following problem:

GIVEN:

(i) An $(n+1)$-dimensional subspace $V \subset C[a,b]$ of function ψ continuous on $[a,b]$, and

(ii) $s_0 < s_1 < \cdots < s_n \in [a, b]$.

For each experiment $\{(s_0, y_0), \cdots, (s_n, y_n)\}$

FIND:

That function $\psi \in V$ which has the property

$$(s_j, y_j) = (s_j, \psi(s_j)), \quad \text{for } j = 0, 1, \cdots, n$$

i.e. the graph of ψ is supposed to pass thru the data points, Eq(*) on page 8A.1

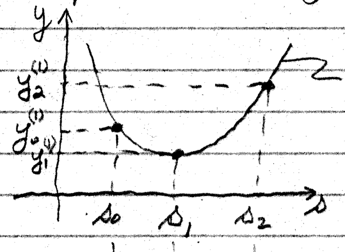
(GO TO P 8A.3)

SOLUTION for $V = \mathcal{P}_2 = \{\psi : \psi(s) = a_0 + a_1 s + a_2 s^2;\ a_i \in R\}$

1. Consider the following evaluation ("sampling") maps $\underset{\sim}{\omega}^0, \underset{\sim}{\omega}^1, \underset{\sim}{\omega}^2 \in V^*$:

$$\langle \omega^0 | \psi \rangle \equiv \omega^0(\psi) = \psi(s_0)$$
$$\langle \omega^1 | \psi \rangle \equiv \omega^1(\psi) = \psi(s_1)$$
$$\langle \omega^2 | \psi \rangle \underset{\text{notation}}{\equiv} \omega^2(\psi) = \psi(s_2)$$

Example 1 (only 3 sample points)

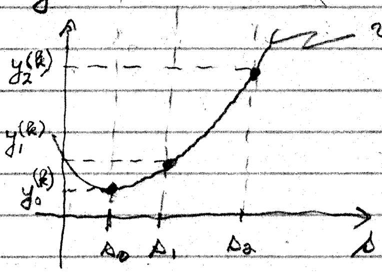


$\psi^{(1)} \in V = P_2$

Result of 1st sampling experiment:

$\{y_0^{(1)}, y_1^{(1)}, y_2^{(1)}\}$

$\boxed{\psi^{(1)} = ??}$



$\psi^{(k)} \in V = P_2$

Result of k^{th} sampling experiment:

$\{y_0^{(k)}, y_1^{(k)}, y_2^{(k)}\}$

$\boxed{\psi^{(k)} = ??}$

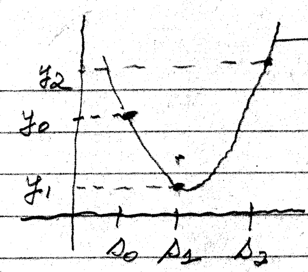
All sampling experiments are performed

at the same s-values, namely,

$\{s = s_0, s = s_1, s = s_2\}$

The result of the k^{th} sampling run is

$\{y_0^{(k)}, y_1^{(k)}, y_2^{(k)}\}$



$\psi \in P_2$

$\boxed{\psi = ??}$

Problem:

For any sampling run whose results is

$\{y_0, y_1, y_2\}$,

FIND $\psi = a_0 + a_1 s + a_2 s^2 \in P_2$

such that

$$\psi(s_0) = y_0$$
$$\psi(s_1) = y_1$$
$$\psi(s_2) = y_2$$

SOLUTION (in 3 steps)

Step 1. Consider the following three evaluation ("sampling") maps $\underset{\sim}{\omega}^0, \underset{\sim}{\omega}^1, \underset{\sim}{\omega}^2 \in V^*$ (the space dual to P_2)

$$\langle \omega^0 | \psi \rangle \equiv \omega^0(\psi) = \psi(s_0)$$
$$\langle \omega^1 | \psi \rangle \equiv \omega^1(\psi) = \psi(s_1)$$
$$\langle \omega^2 | \psi \rangle \equiv \omega^2(\psi) = \psi(s_2)$$

Dirac Notation

They constitute a linearly indendent set (=basis) for V^*

$$B^* = \{\underset{\sim}{\omega}^0, \underset{\sim}{\omega}^1, \underset{\sim}{\omega}^2\} \subset V^* = (\mathcal{P}_2)^*$$

Step2. The solution to this example problem consists of finding the basis

$$B = \{\psi_0(s), \psi_1(s), \psi_2(s)\} \subset \mathcal{P}_2 \text{ for } V = \mathcal{P}_2.$$

which satisfies the duality principle ("duality relation")

$$\boxed{\langle \omega^j | \psi_i \rangle \equiv \omega^j(\psi_i) = \psi_i(s_j) = \delta^j{}_i}$$

Comment: Notice that here in step2 we are asked to execute the converse of Theorem 9 on page 8.5, namely, namely given

$$B^* = \{\underset{\sim}{\omega}^1, \ldots, \underset{\sim}{\omega}^n\} \text{ for } V^*,$$

construct vectors

$$\{\vec{e}_1, \ldots, \vec{e}_n\} \equiv B$$

such that $\langle \underset{\sim}{\omega}^j | \vec{e}_i \rangle = \delta^j{}_i$.

~~They constitute a lin. indep. spanning set for V^*; i.e. they form a basis~~

~~$B^* = \{\omega^0, \omega^1, \omega^2\}$ for $V^* = \mathcal{P}_2^*$~~ See HW#2 problem 8

~~2. Let $B = \{\psi_0, \psi_1, \psi_2\} \subset \mathcal{P}_2$ be the~~ for $V = \mathcal{P}_2$

~~Such a basis satisfies~~

~~$\delta^i_j = \langle \omega^i | \psi_j \rangle \equiv \omega^i(\psi_j) = \psi_j(s_i)$~~

CONTINUED FROM P8A.5

The explicit formulae for these basis elements have been exhibited by the Enlightenment mathematician Lagrange; they are

$$\psi_0(s) = \frac{(s-s_1)(s-s_2)}{(s_0-s_1)(s_0-s_2)}$$

$$\psi_1(s) = \frac{(s-s_0)(s-s_2)}{(s_1-s_0)(s_1-s_2)}$$

$$\psi_2(s) = \frac{(s-s_0)(s-s_1)}{(s_2-s_0)(s_2-s_1)}$$

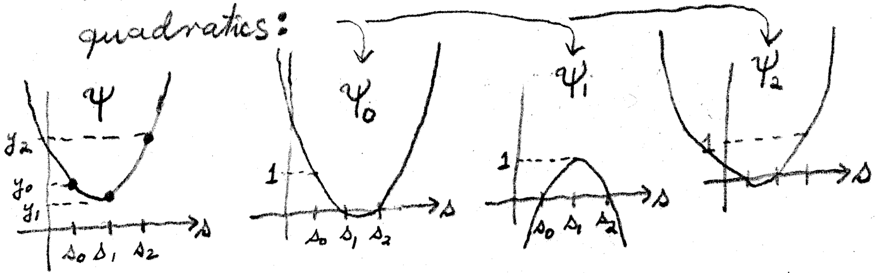
They satisfy the required duality relation

$$\psi_j(s_i) = \delta_j{}^i = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Step 3. The sought-after function ψ

$$\boxed{\psi(s) = y_0\,\psi_0(s) + y_1\,\psi_1(s) + y_2\,\psi_2(s)} \quad (**)$$

is a quadratic whose graph passes through the data points, Eq. (*) near the top of page 8A.1, It is the a linear combination of three simple quadratics:



$$\psi \quad = \quad y_0\psi_0 \quad + y_1\psi_1 \quad + y_2\psi_2$$

And as mentioned on the previous page, these three quadratic are basis vectors for $V = \mathcal{P}_2$ dual to the three basis covectors $\omega^0, \omega^1, \omega^2$ for V^*:

$$\left.\begin{aligned} \omega^i(\psi_0) &\equiv \psi_0(s_i) = \delta_{i0} \\ \omega^i(\psi_1) &\equiv \psi_1(s_i) = \delta_{i1} \\ \omega^i(\psi_2) &\equiv \psi_2(s_i) = \delta_{i2} \end{aligned}\right\} \text{Look at the above graphs}$$

Summary: Eq. (**) at the top of this page constitutes the solution to sample prolem 1.

Example 2

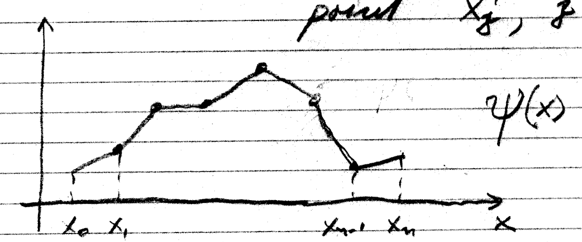
V = space of continuous piece-wise linear functions on the closed interval $[x_0, x_n]$

$= CPL(\{x_0, \ldots, x_n\})$

where

$$x_0 < x_1 < x_2 \cdots < x_n$$

and $\psi \in V$ is linear in each of the subintervals $[x_0, x_1], \ldots, [x_{n-1}, x_n]$ and continuous at each point x_j, $j = 0, 1, \ldots, n$



Claim: V is a vector space?

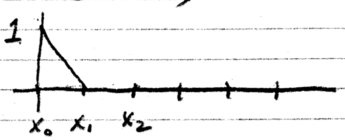
Consider the basis $\{\psi_i : i = 0, \ldots, n\} \subset V$ with the property

$$\psi_i(x_j) = \delta^j_i \quad (\equiv \langle \omega^j \mid \psi_i \rangle)$$

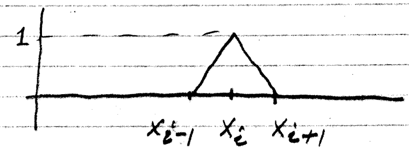
"the jth evaluation map applied to the vector ψ_i"

Example 2 (cont'd)

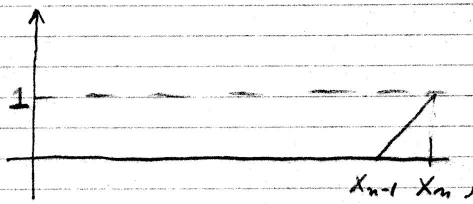
$\psi_0(x) \in V$:



$\psi_i(x) \in V$:



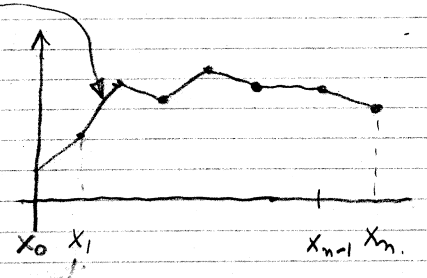
$\psi_n(x) \in V$:



$$\boxed{\langle \omega^j \mid \psi_i \rangle \equiv \psi_i(x_j) = \delta_i^j}$$

$\psi_i(x)$ $i = 0, 1, \ldots, n$ are the unit hat functions

$\psi = \sum_{i=0}^{n} y_i \psi_i(x)$ where $y_i = \psi(x_i)$



$V = \text{span}(\{\psi_i\})$

$V^* = \text{span}(\{\underset{\sim}{\omega}^j\})$

Comment

Let $\underset{\sim}{f}$ = a weighted sampling (evaluation) map with weights ξ_i

Applying this kind of sampling map to $\psi = \sum_j y_j \psi_j \in V$ one obtains

$$\underset{\sim}{f}(\psi) = f\left(\sum_j y_j \psi_j\right)$$

$$= \sum_j f(\psi_j)\, y_j$$

$$= \sum_j \underset{\sim}{f}(\psi_j)\, \underset{\sim}{\omega}^j(\psi)$$

Thus

$$\underset{\sim}{f} = \sum_j \underset{\sim}{f}(\psi_j)\, \underset{\sim}{\omega}^j \in V^*$$

in terms of the basis sampling maps $\{\underset{\sim}{\omega}^j\}$:

$$\underset{\sim}{f} = \sum_j \xi_j\, \underset{\sim}{\omega}^j .$$

The "weights"

$$\xi_j = \underset{\sim}{f}(\psi_j) = \langle \underset{\sim}{f} \mid \psi_j \rangle$$

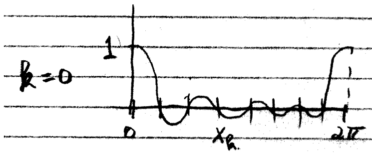
are the coordinates of $\underset{\sim}{f}$ relative to the dual basis $\{\underset{\sim}{\omega}^j\}$ for V^*.

Vector space of band-limited functions

Example 3: For the case of the vector space $V = B_N$ of bandlimited function of period 2π

$$V = \{\psi : \psi(x) = \sum_{m=0}^{N} a_m \cos mx + \sum_{m=1}^{N} b_m \sin mx\} \equiv B_N,$$

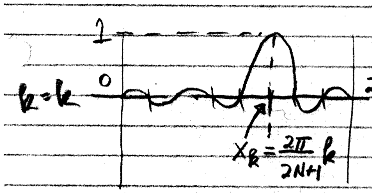
Consider the following set of basis function



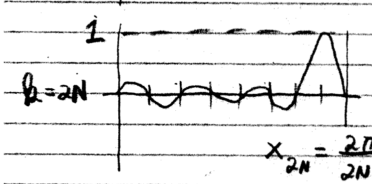
$$\psi_k(x) = \frac{1}{2N+1} \sum_{m=-N}^{N} e^{im(x-x_k)}$$

$$= 1 + \frac{2}{2N+1} \sum_{m=1}^{N} \cos m(x - x_k) \in B_N$$

where $x_k = \frac{2\pi}{2N+1} k$



$$\psi_k(x) = \frac{1}{2N+1} \frac{\sin(N+\frac{1}{2})(x-x_k)}{\sin\frac{x-x_k}{2}}$$



$$\psi_{2N}(x) = \frac{1}{2N+1} \frac{\sin(N+\frac{1}{2})(x-x_{2N})}{\sin\frac{x-x_{2N}}{2}}$$

$x_{2N} = \frac{2\pi}{2N+1} 2N$

They belong to V because

$$\psi_k(x) = \frac{1}{2N+1} \sum_{m=-N}^{N} e^{im(x-x_k)} \overset{\text{geometrical series}}{=} \frac{1}{2N+1} \frac{\sin(N+\frac{1}{2})(x-x_k)}{\sin\frac{x-x_k}{2}}$$

$$= 1 + \frac{2}{2N+1} \sum_{m=1}^{N} \cos m(x - x_k) \in B_N$$

is also bandlimited

These basis functions satisfy

$$\psi_k(x_i) = \delta_{ki} \qquad \text{(why?)}$$

The amazing thing about the bandlimited functions is that, although they are continuous, each one may be reconstructed with 100% accuracy from a set of sampled values:

$$\boxed{\psi(x) = \sum_{k=0}^{2N} \psi(x_k)\, \psi_k(x)}$$

$\psi(x)$ in terms of sampled data $\{\psi(x_k) : k = 0, 1, \dots, 2N\}$

(This is a special case of the Whittaker – Shannon sampling theorem)

Comment: Each of the basis functions is the response of a low pass filter to a periodic train of impulses