LECTURE 9

Bilinear functional

Metric as an Inner product

Metric as a Natural isomorphism between V and V^*

The vector space arenas developed so far are in sceletal form but fundamental to all of mathematics. Their linearity is captured by means of the superposition principle. The bare bones structures introduced so far are linear (in)dependence and the spanning property of a set of vectors. These structures are building blocks sufficient for characterizing a vector space in terms of coordinate systems introduced via any chosen (or given) basis.

Furthermore, every vector space (V) always

accommodates its dual space, the space of linear function. This space (V^*) is a vector space in its own right, and any given basis for V determines a unique corresponding basis for V^*.

Indeed the dimension of V and V^* are the same, a fact which is a consequence of the duality principle

$$\langle \underset{\sim}{\omega}^j | \vec{e}_i \rangle = \delta^j_i .$$

In spite of this, there is no natural (i.e. basis-independent) correspondence between V and its dual space of covectors, V^*.

This deficiency, as we shall see, disappears once one has identified an inner product on the given vector space arena. Once one knows the rule for inner products between vectors, then this inner product rule furnishes a natural correspondence between V and V^*.

An <u>inner product</u> on V is called a <u>metric structure</u> on V, and hence on V^*. We shall do this in both (i) a basis independent way and in (ii) a way which takes advantage a chosen basis.

BILINEAR FUNCTIONAL. THE METRIC

There is no natural isomorphism between V and V^*. However if the vector space has an inner product defined on it, then such an isomorphism is determined. ~~The manner~~ An inner product is implemented on the vector space V by means of the following set of definitions.

Definition 7 (Bilinear Form)

Given: a vector space U and a vector space V.

A bilinear functional (or "form") on $U \times V$ (pairs of elements, one from U and one from V) is a function w;

$$w: U \times V \xrightarrow{w} \text{reals}$$
$$(x, y) \rightsquigarrow w(x,y)$$

with the properties

$$w(\alpha^1 x_1 + \alpha^2 x_2, y) = \alpha^1 w(x_1, y) + \alpha^2 w(x_2, y)$$
$$w(x, \beta^1 y_1 + \beta^2 y_2) = \beta^1 w(x, y_1) + \beta^2 w(x, y_2);$$

in other words, w is linear in each argument.

Definition 8 (Metric)

A metric (or inner product) is a bilinear functional g on $V \times V$ (pairs of elements in V)

$$g: (x,y) \rightsquigarrow g(x,y)$$

with the property
$$g(x,y) = g(y,x);$$

in other words, a real valued metric is symmetric. ~~If the~~

Comment: If the metric were complex valued then the symmetry condition gets replaced by
$$g(x,y) = \overline{g(y,x)}$$

If $g(\ ,\ )$ is a scalar product provided g is positive definite, i.e $g(x,x) > 0 \ \forall x \neq 0$

In our development of tensor algebra we shall not insist that g be positive definite. This means that ~~we~~ allow for the existence of non-zero vectors such that
$$g(x,x) = 0, \text{ with } x \neq 0.$$

These vectors are "null-vectors". ~~We~~ We are forced to consider an inner product with such a null result if V is Minkowski spacetime.

Euclidean Space & Minkowski Spacetime

Example (Basis expansion of the metric)

Let $\vec{X} = x^1\vec{e}_1 + x^2\vec{e}_2 + \cdots + x^n\vec{e}_n$ be a representation of a vector X in terms a basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$ for V. Then

$$g(x^1e_1 + x^2e_2 + \cdots, y^1e_1 + y^2e_2 + \cdots) =$$

$$= x^1y^1\, g(e_1, e_1) + (x^2y^1 + x^1y^2)\, g(e_1, e_2) + x^2y^2 g(e_2, e_2) + \cdots$$

$$\equiv x^1y^1\, e_1\cdot e_1 + (x^2y^1 + x^1y^2)\, e_1\cdot e_2 + x^2y^2\, e_2\cdot e_2 + \cdots$$

$$\equiv x^1y^1 g_{11} + (x^2y^1 + x^1y^2)\, g_{12} + x^2y^2 g_{22} + \cdots$$

$= x^i y^j g_{ij}$ (Einstein summation convention for pairs of rep'd indices)

The coefficients $g_{ij} \equiv e_i\cdot e_j \equiv g(e_i, e_j)$

are the components of the metric g with respect to to the basis $\{e_1, e_2, \ldots, e_n\}$

They are the inner products of all pairs of basis vectors.

~~2.1 Metric as a Map Between Vectors and Covectors. The scalar product is a bilinear function. Consequently, it can be evaluated on only one of its arguments. The result is obviously a linear function. More precisely,~~

2.1 Metric as a Map between Vectors and Covectors.

The scalar product

$$g: V\times V \to R$$
$$(x, y) \mapsto g(x, y)$$

is a bilinear function. Consequently it can be evaluated on only one of its arguments, $g(x, \ )$. The result is a linear function. More precisely, a metric establishes a natural isomorphism between vectorspace V and its space of duals, V^*. In order to conserve notation we shall use the same symbol g to designate this correspondence. Its defining property is

$$g: V \xrightarrow{g} V^*$$
$$x \rightsquigarrow g(x, \ ) = \underset{\sim}{x} \quad (= "x\cdot")$$

Here $\underset{\sim}{x}$ is that linear functional which, when operating on $y \in V$, yields $g(x,y)$:

$$\underset{\sim}{x} = "x\cdot": \quad V \xrightarrow{\underset{\sim}{x}} R$$
$$y \rightsquigarrow \langle \underset{\sim}{x} | y \rangle = x\cdot y \equiv g(x,y)$$

GO to TOP 9.9

From Bottom of P21.5

~~One can use the representation of the metric relative to a given basis,~~

$$\cancel{g = g_{ij}\, \underset{\sim}{\omega}^i \otimes \underset{\sim}{\omega}^j}$$

~~to implement the isomorphism $V \to V^*$ on the components of the vectors in V and the corresponding covectors in V^*. This is done as follows;~~

Proposition (How to explicitly relate vectors in V to those in V^*)

Given $\vec{x} = x^k e_k$, the numerical coefficients x_j of $\underset{\sim}{x} = x_j \underset{\sim}{\omega}^j$ are given explicitly by the following computation

$$\underset{\sim}{x} = g(\underbrace{\vec{x}}_{x^k e_k}, \ ) = (??)_j\, \underset{\sim}{\omega}^j \equiv \underbrace{x_j}_{?}\, \omega^j$$
$$= x^k g(e_k, \ )$$

$$\therefore \quad x^k g(e_k, e_i) = x_j \underbrace{(\omega^j | e_i)}_{\delta^j_i}$$

or

$$x^k g_{ki} = x_i$$

In other words,

$$\boxed{x_i = g_{ik} x^k} \boxed{= \vec{x}\cdot e_i}$$

are the components of $\underset{\sim}{x} = \omega^j x_j$, which is the image of $x = x^k e_k$ under g.

~~GO BACK TO P 21.4~~

One can use the representation of the metric relative to a given basis,

$$g = g_{ij}\, \omega^i \otimes \omega^j$$

to implement the isomorphism $V \to V^*$ between the components of vectors in V and the components of covectors in V^*. This is done as follows.

Proposition: (Basis representation of g)

Let $\{e_i\}$ be a basis of V
Let $\{\omega^j\}$ be its dual basis for V^*

Then g can be written in terms of a "tensor product" basis as follows:

$$g = g_{ij}\, \underset{\sim}{\omega}^i \otimes \underset{\sim}{\omega}^j : V \longrightarrow V^*$$

$$x \rightsquigarrow g(x,\ ) = g_{ij} \underbrace{\langle \underset{\sim}{\omega}^i | x \rangle}_{\vec{x}\cdot e_j} \underset{\sim}{\omega}^j$$

$$e_i \rightsquigarrow g(e_i,\ ) = g_{ij}\, \underset{\sim}{\omega}^j$$

(image of e_i in V^* relative to basis $\{\underset{\sim}{\omega}^j\}$)

Comment:

1. Here the tensor product sign $\otimes$ establishes an ordered juxtaposition of pairs of linear functionals. This is the means by which a bilinear functional is constructed from a pair of linear functionals. The tensor product is therefore a means of obtaining a bilinear functional from two linear functionals.

2. The "components" of g relative to e_k and ω^l are obtained by evaluating it on pairs of vectors

$$g(x,y) = g_{ij}\langle \omega^i | x\rangle \langle \omega^j | y \rangle = g_{ij}\, x^i y^j$$

$$g(e_k, e_l) = g_{ij}\langle \omega^i | e_k\rangle \langle \omega^j | e_l \rangle$$

$$= g_{ij}\, \delta^i_k\, \delta^j_l$$

$$= g_{kl} \quad (= g_{lk})$$

(over)

More generally, evaluating g on the pair of vectors $x, y \in V$, one obtains their inner product

$$x \cdot y = g(x^k e_k, y^l e_l) = g_{kl} x^k y^l$$
$$= x_l y^l$$