TENSOR CALCULUS 10

10. Metric

10.1 Metric-Induced Parallel Transport

10.2 Metric Compatibility via Extremal Paths

Proverb of the Day

The existence of a variational integral for gravitation leads to a "metrical elasticity" of space, i.e. to generalized forces which oppose the curving of space.

Andrei D. Sakharov

Sov. Phys. Doklady 12, 1040–1041 (May 1968)

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5. Metric

Parallel transport establishes a correspondence between vectors in the vector space at one point and those in the vector space at another point. A pair of corresponding vectors are decreed to be parallel by this law of parallel transport.

Parallel transport makes no statements about angles between vectors or their length. Such statements are in the purview of a different geometrical structure, namely that of a metric (tensor) on a vector space, or more generally, a metric tensor field, which is an assignment of a metric tensor, say

$$g \equiv \text{"}\cdot\text{"} \equiv ds^2 = g_{ij}\, \omega^i \otimes \omega^j$$

to each point $\mathcal{P}$ of the manifold. Here

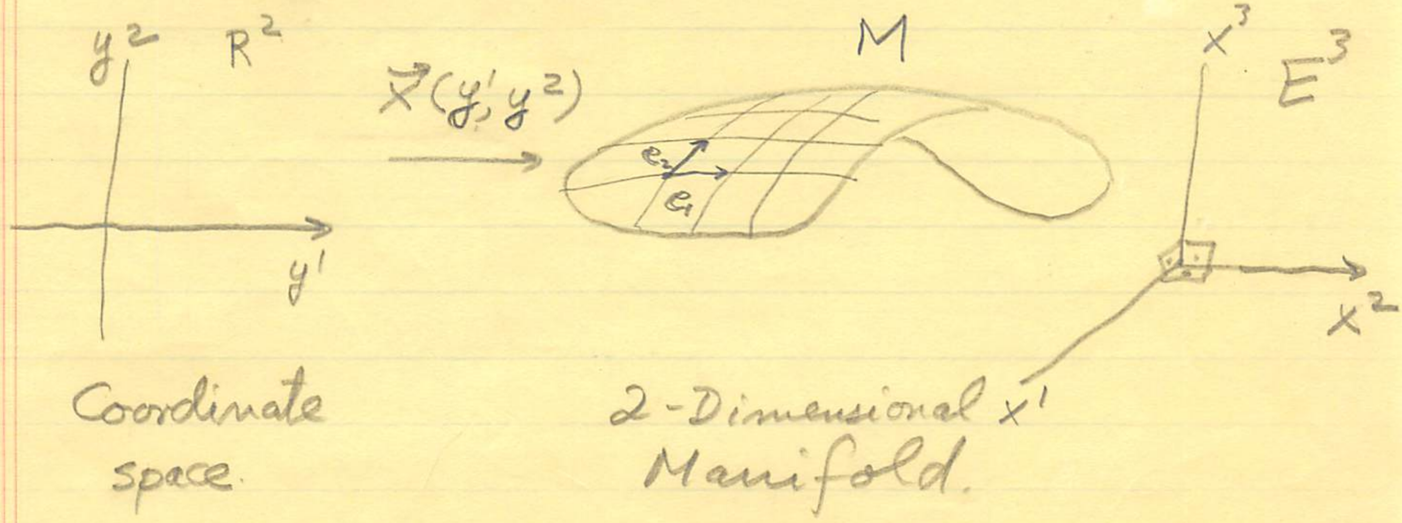
$$g_{ij} = e_i \cdot e_j$$

are the inner products of the basis vectors for $V_{\mathcal{P}}$.

Let us consider two examples, one from geometry, the other from continuum mechanics.

First Example.
(Metric induced on a 2-dimensional surface by the geometry of the ambient Euclidean space)

In Euclidean 3 dimensional space consider a surface parametrized by the coordinates y^1 and y^2:



Coordinate space

2-Dimensional Manifold.

The tangent space V_P at each point of M is spanned by the coordinate basis

$$\left\{ e_1 = \frac{\partial}{\partial y^1}, e_2 = \frac{\partial}{\partial y^2} \right\}$$

The partial derivatives

$$\left\{ \frac{\partial x^1}{\partial y^i}, \frac{\partial x^2}{\partial y^i}, \frac{\partial x^3}{\partial y^i} \right\} \longleftrightarrow \frac{\partial \vec{x}}{\partial y^i} \qquad i = 1, 2$$

are the components of the vectors e_1 and e_2 relative to the orthonormal coordinates of the ambient three dimensional Euclidean space. This Euclidean space quite naturally <u>induces</u> the following <u>inner product</u> on the tangent space V_P at the point P of M: One has

Metric induced covariant derivative (parallel transport)

$$e_i \cdot e_j = \left(= \frac{\partial}{\partial y^i} \cdot \frac{\partial}{\partial y^j}\right) = \sum_{m=1}^{3} \frac{\partial x^m}{\partial y^i} \frac{\partial x^m}{\partial y^j} \left(= \frac{\partial \vec{x}}{\partial y^i} \cdot \frac{\partial \vec{x}}{\partial y^j}\right)$$

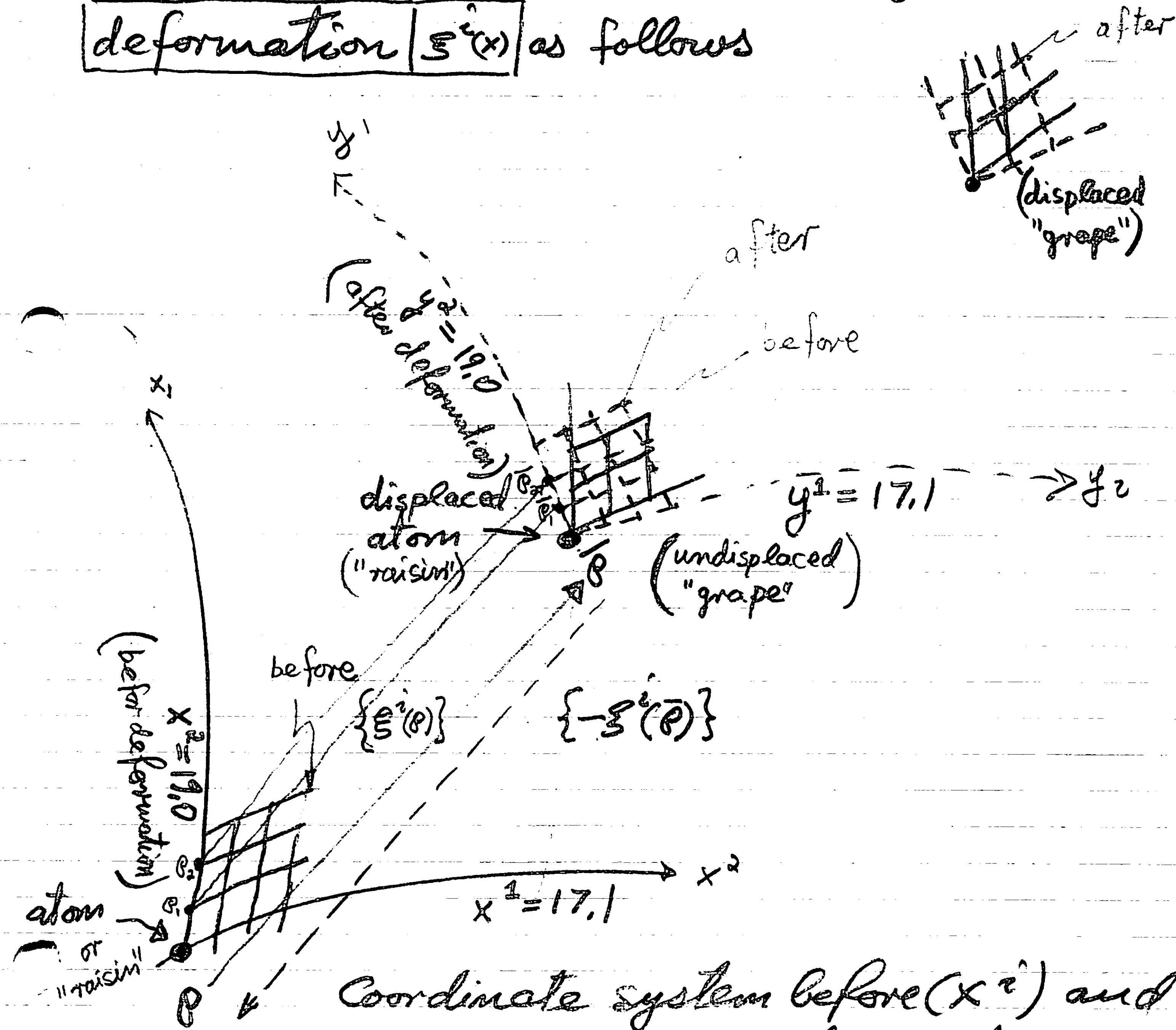
Thus the metric tensor on this imbedded manifold is

$$g = \frac{\partial \vec{x}}{\partial y^i} \cdot \frac{\partial \vec{x}}{\partial y^j} \; dy^i \otimes dy^j = g_{ij}\, dy^i \otimes dy^j$$

Thus the <u>inner product on E^3 induces</u> a natural <u>metric tensor field</u> on the imbedded submanifold M indeed.

Second Example.
(strain metric induced in an elastic medium by a deformation away from equilibrium)

Consider an elastic medium subjected to a $\boxed{\text{deformation}\ \boxed{\xi^i(x)}}$ as follows



Coordinate system before (x^i) and after (y^i) the deformation

Evidently the effect of the deformation is two-fold: (i) it changes the location of each atom ("raisin" in a gelatinous medium) and (ii) it changes the curvilinear coordinate system used to map out the atoms.

Here is the analysis:

The deformation (say F), is a map from the medium onto itself : $M \to M$

$$P \mapsto \bar{P} = F(P)$$

Consider a curvilinear coordinate system $x^i(P)$ attached to the atoms of the medium:

$P \overset{F}{\mapsto} \bar{P}$ "raisin coordinate"   Displaced point object

num. values

$\boxed{x^i(P)}$	$\boxed{y^i(\bar{P}) = x^i(P)}$ new at new pt = old at old pt	$\boxed{x^i(\bar{P}) = x^i(P) + \xi^i(P)}$
Given	Definition of new coordinate system induced by the given deformation F.	Definition of ξ^i, the component of the strain displacement vector $x^i(P) = x^i(\bar{P}) - \xi^i(P)$

We assign old coord at old point to new coord. at new point.

Hence the coordinate values of the old (x^i) and the new (y^i) coordinates are related by the strain function ξ^i

$$x^i(\bar{P}) - y^i(\bar{P}) = \xi^i(P) \quad \begin{pmatrix}\text{old grape}\\ \text{coordinate}\end{pmatrix} - \begin{pmatrix}\text{new}\\ \text{raisin}\\ \text{coord}\end{pmatrix} = \begin{pmatrix}\text{strain}\\ \text{displacement}\end{pmatrix},$$

Drop the bar to obtain

$$= \xi^i(\bar{P} + P - \bar{P}) = \xi^i(\bar{P}) + \underbrace{\frac{\partial \xi^i}{\partial x^j}(-)\xi^j + \text{etc}}_{\text{neglect}}$$

$$\boxed{y^i(x^j) = x^i - \xi^i(x^j)} \quad \approx \xi^i(\bar{P})$$

Thus, once the strain displacement $\{\xi^i(x)\}$ is given, the new coordinate system is determined.

The deformation $\mathcal{P} \overset{E}{\rightsquigarrow} \bar{\mathcal{P}}$ establishes the map $V_{\mathcal{P}} \longrightarrow V_{\bar{\mathcal{P}}}$. It does this by mapping the old basis vectors at $\mathcal{P}$ to the new <u>distorted</u> basis at $\bar{\mathcal{P}}$. This distortion can be a shear, a volume change, or any combination of such changes in the basis vectors. The change in the inner product between any pair of distorted basis vectors expresses such a distortion. In fact,

$g_{old} = g_{ij}(x(\mathcal{P}))\, dx^i dx^j$ is the old inner product for $V_{\mathcal{P}}$ while the <u>new</u>, <u>distorted</u> <u>inner product</u> at $V_{\mathcal{P}}$ is

$$g_{ij}(y(\mathcal{P}))\, dy^i \otimes dy^j = g_{ij}(x^k - \xi^k)\, d(x^i - \xi^i) \otimes d(x^j - \xi^j)$$

keep only the principal linear part:

$$= g_{ij}(x)\, dx^i dx^j - g_{ij,k}\, \xi^k dx^i dx^j$$

$$- g_{ij}\, \xi^i_{\ ,k}\, dx^k dx^j - g_{ij}\, \xi^j_{\ ,k}\, dx^i dx^k$$

$$\boxed{g_{NEW} = g_{ij}\, dx^i \otimes dx^j - \underbrace{\left(\frac{\partial g_{ij}}{\partial x^k} \xi^k + g_{kj} \frac{\partial \xi^k}{\partial x^i} + g_{ik} \frac{\partial \xi^k}{\partial x^j} \right) dx^i \otimes dx^j}_{S_{ij}\, dx^i \otimes dx^j \,=\, \text{strain tensor};}}$$

$+$ terms of higher order in ξ

One sees that the distortion, (i.e. perturbation) in the metric, the strain tensor

$$S_{ij}\,dx^i\otimes dx^j = \left(g_{ij,k}\,\xi^k + g_{kj}\,\xi^k{}_{,i} + g_{ik}\,\xi^k{}_{,j} \right) dx^i \otimes dx^j$$

(a) is symmetric,

and
$$S_{ij} = S_{ji}\,;$$

(b) relative to a linear (as compared to a curvilinear) coordinate system, for which $g_{ij} = \text{const.}$ so that $g_{ij,k} = 0$, one has

$$\boxed{S_{ij} = \xi_{j,i} + \xi_{i,j}}$$

where we "lowered" the index of ξ^k by means of

$$g_{kj}\,\xi^k{}_{,i} = (g_{kj}\,\xi^k)_{,i} \equiv \xi_{j,i}$$

End of example.

Reminder: Vectors, or more generally tensors, can only be added provided they are located at the same point P, i.e. lie in the same tangent space.

28.10

5.1 Metric-Induced Parallel Transport

Given a metric tensor field

$$g_{ij}\, \underset{\sim}{\omega}^i \otimes \underset{\sim}{\omega}^j$$

one finds that there is in essence only one law of parallel transport

$$\boxed{\begin{aligned} d\mathbf{e}_i &= \mathbf{e}_j\, \underset{\sim}{\omega}^j{}_i \\ &\equiv \mathbf{e}_j\, \Gamma^j{}_{ik}\, \underset{\sim}{\omega}^k \\ \nabla_k \mathbf{e}_i &= \mathbf{e}_j\, \Gamma^j{}_{ik} \end{aligned}}$$

$\langle \underset{\sim}{d}\mathbf{e}_i | \mathbf{e}_k \rangle$

which is compatible with this given metric; in other words, the Christoffel symbols $\Gamma^j{}_{ik}$ are uniquely determined once the metric coefficient functions $g_{ij}(x)$ are known.

More precisely, we have the following

<u>Proposition</u> (metric induced parallel transport)
[$\{g, T=0\} \Rightarrow \exists$ exists a unique ∥xport]

$g, T=0$ ⇓ unique ∥xport.

On a manifold without torsion, g implies a unique parallel transport, i.e. covariant differentiation ~~(determined by metric tensor field and commutators of basis vectors)~~

This proposition follows from the <u>requirement</u> that the point to point change of the inner product of basis vectors,

$$g_{ij} = e_i \cdot e_j$$

is such that

(1) if $\qquad g_{ij} = \text{const}$ then $\Gamma^i{}_{jk} = 0$

i.e. the vectors have been chosen parallel. But if (2) $g_{ij} \neq \text{const}$, then its change is only due to their <u>non-parallelness</u>, i.e. the change must be one corresponding to the <u>difference of products</u>,

$$\boxed{dg_{ij} \equiv d(e_i \cdot e_j) = \underset{\sim}{d}e_i \cdot e_j + e_i \cdot \underset{\sim}{d}e_j} \quad (28.12a)$$

In terms of the partial derivatives ∇_k this equation reads

$$\boxed{\nabla_k g_{ij} \equiv \nabla_k(e_i \cdot e_j) = (\nabla_k e_i)\cdot e_j + e_i \cdot (\nabla_k e_j)} \quad (28.12b)$$

Equation 28.12 is the <u>compatibility</u> condition which the law of parallel transport

$$\underset{\sim}{d}\, e_i = e_\ell\, \underset{\sim}{\omega}^\ell{}_i$$

$$\equiv e_\ell\, \Gamma^\ell{}_{ik}\, \underset{\sim}{\omega}^k,$$

or

$$\nabla_k e_i = e_\ell\, \Gamma^\ell_{ik},$$

must satisfy. It is merely a matter of algebra to see how the Γ^ℓ_{ik} are determined by the partial derivatives, which we shall abreviate by

$$\nabla_k g_{ij} \equiv g_{ij,k}\,.$$

Indeed, the <u>compatibility</u> <u>condition</u> yields

$$g_{ij,k} = (\nabla_k e_i)\cdot e_j + e_i \cdot (\nabla_k e_j)$$

$$= \Gamma^{l}_{ik}\, e_l \cdot e_j + \Gamma^{l}_{jk}\, e_i \cdot e_l$$

$$= g_{lj}\,\Gamma^{l}_{ik} + g_{il}\,\Gamma^{l}_{jk} \qquad (28.13)$$

Before we proceed to solve these equations for Γ^{l}_{ik}, it is appropriate to take note of its geometrical significance.
Recall that the covariant derivative of a tensor of rank $\binom{0}{2}$ is given by

$$\nabla_k \underset{\sim}{g} = \nabla_k (g_{ij}\, \underset{\sim}{\omega}^i \otimes \underset{\sim}{\omega}^j) = g_{ij;k}\, \underset{\sim}{\omega}^i \otimes \omega^j$$

$$= (\nabla_k g_{ij})\, \omega^i \otimes \omega^j + g_{ij} \underbrace{(\nabla_k \omega^i)}_{-\Gamma^i_{lk}\,\omega^l}\otimes \omega^j + g_{ij}\, \omega^i \otimes \nabla_k \omega^j$$

where the components of this derivative are

$$g_{ij;k} = g_{ij,k} - g_{lj}\,\Gamma^{l}_{ik} - g_{il}\,\Gamma^{l}_{jk}$$

Comparing them with the compatibility condition yields,

$$g_{ij;k} = 0$$

In other words,

$$\boxed{\nabla_k g = 0} \qquad k = 1, \ldots, n.,$$

the compatibility between the metric tensor field and the parallel transport it induces is equivalent to the geometrical statement that the metric is covariant constant.

We now return to the task of solving Eq. (28,13) for the coefficients $\Gamma^{\ell}{}_{ik}$ in terms of the derivatives $g_{ij,k}$. To do this we introduce for the sake of efficiency

$$\Gamma_{m\,ik} = g_{m\ell}\,\Gamma^{\ell}{}_{ik},$$

which used to be known as the Christoffel symbols of the "first kind."

In terms of these the compatibility condition, Eq. (28.13), assumes the simple form

$$g_{ij,k} = \Gamma_{jik} + \Gamma_{ijk}.$$

A key ingredient to the uniqueness of the coefficients Γ^{ℓ}_{ik} is that the the parallel transport be torsionless, as was stipulated in the statement of the proposition. Consequently,

$$T=0 \Rightarrow \quad \nabla_k e_i - \nabla_i e_k = \underset{+\mathbb{T}(e_k, e_i)}{[e_k, e_i]} \equiv \underset{+T^{\ell}_{ki} e_\ell}{c_{ki}{}^{\ell} e_\ell}$$

or

$$\Gamma^{\ell}{}_{ik} - \Gamma^{\ell}{}_{ki} = c_{ki}{}^{\ell} + \text{zero}.$$

What does this restriction tell us? The answer is particularly perspicuous relative to a <u>coordinate</u> basis, relative to which $c_{ki}{}^{\ell} = 0$. Thus, the zero torsion stipulation manifests itself by the condition

$$\Gamma^{\ell}_{ik} = \Gamma^{\ell}_{ki} \qquad \text{(coordinate basis only)}$$

namely that the Christoffel symbols are <u>symmetric</u> in the last two indeces.

~~But~~ For a general basis one has

$$\Gamma^{\ell}_{ik} = \tfrac{1}{2}\left(\Gamma^{\ell}_{ik} + \Gamma^{\ell}_{ki}\right) + \tfrac{1}{2}\, c_{ki}{}^{\ell}$$

This is the familiar decomposition of a two index (i and k) object into a part which is symmetric and another part which is antisymmetric under the interchange $i \leftrightarrow k$.

We also note in passing that the commutator coefficient $c_{ki}{}^{\ell}$ has its subscripts backward from those of Γ^{ℓ}_{ik}.

4. We wish to invert the equation

$$g_{ij,k} = \overset{①}{\Gamma}_{i\,jk} + \Gamma_{j\,ik}\,; \quad \Gamma_{ijk} = \Gamma_{ikj}\,;\ \Gamma_{jik} = \Gamma_{jki}$$

Using $\Gamma_{ijk} - \Gamma_{ikj} = g_{il}\, c_{kj}{}^{l} \equiv c_{kji}$

$$\Gamma_{i\,jk} = \tfrac{1}{2}\left(\overset{①}{\Gamma}_{i\,jk} + \overset{2}{\Gamma}_{j\,ik}\right) + \tfrac{1}{2}\left(\overset{①}{\Gamma}_{i\,kj} + \overset{1}{\Gamma}_{k\,ij}\right)$$

"cancel" cancel

$$- \tfrac{1}{2}\left(\Gamma_{j\,ki} + \Gamma_{k\,ji}\right) + \tfrac{1}{2}\left(\overset{3}{c}_{kj\,i} - \overset{2}{c}_{ki\,j} - \overset{1}{c}_{ji\,k}\right)$$

which becomes

$$\Gamma_{i\,jk} = \tfrac{1}{2}\left(g_{ij,k} + g_{ik,j} - g_{jk,i}\right)$$

$$+ \tfrac{1}{2}\left(\overset{1}{c}_{ij\,k} + \overset{2}{c}_{ik\,j} - \overset{3}{c}_{jk\,i}\right)$$

Thus we see that all the connection coefficients are determined by the metric tensor field and the commutators of the basis vectors.

This completes the proof of the proposition.

5. One rarely uses the general formula for Γ_{ijk}

a) Often one chooses an orthonormal basis. In that case

$$g_{ij} = \text{const} = 0,\ 1,\ -1$$

In this case all derivatives of g_{ij} vanish.

OMIT

One has
$$\Gamma_{ijk} = -\Gamma_{jik},$$
or, in terms of
$\Gamma_{ijk}\underset{\sim}{\omega}^k \equiv g_{il}\Gamma^l{}_{jk}\underset{\sim}{\omega}^k \equiv g_{il}\underset{\sim}{\omega}^l{}_j \equiv \underset{\sim}{\omega}_{ij}$ ("rotation 1-forms")
$$\boxed{\underset{\sim}{\omega}_{ij} = -\underset{\sim}{\omega}_{ji}}$$
(An O.N. basis is characterized by pure rotation, no expansion or contraction of its basis vectors as one moves from point to point)

In that case the $\underset{\sim}{\omega}_{ij}$ generate a pure rotational change in the bases assigned to different point.

b) More often yet, one chooses a coordinate bases
$$e_i = \frac{\partial}{\partial x^i}$$
so that $c_{[ij]l} = 0$
$$\Gamma_{ljk} = \Gamma_{lkj}$$
and
$$g^{il}\Gamma_{ljk} = \Gamma^i{}_{jk} = \Gamma^i{}_{kj} =$$

In that case
$$\Gamma^i{}_{jk} = \frac{1}{2}\, g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right) \equiv \left\{ {i \atop jk} \right\}$$
is used to compute the connection coefficients which in this case are also called the <u>Christoffel symbols</u> and are also written as $\left\{ {i \atop jk} \right\}$.

5.2 Metric Compatibility via Extremal Paths

A very direct way of obtaining the metric compatible law of parallel transport (i.e. the coefficients $\Gamma^i{}_{jk}$) is via the extremal paths, the geodesics, determined by the metric. Recall that these extremal paths are obtained from the variational principle

$$\int_{\lambda_1}^{\lambda_2} \sqrt{g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}} \, d\lambda = \text{extremum}$$

The Euler-Lagrange equation for the variational integral reads

$$\frac{d^2x^i}{ds^2} + \frac{dx^l}{ds}\frac{dx^k}{ds}\Gamma^i_{lk} = 0 \qquad (28.19)$$

These Christoffel symbols, relative to a coordinate basis, are furnished directly by this variational principle.

The fact that

$$\Gamma^{i}{}_{\ell k} = \frac{1}{2} g^{ij}\left(g_{j\ell,k} + g_{jk,\ell} - g_{\ell k,j}\right)$$

is guaranteed by the E-L equation, as we saw in Section 6.T of Chapter I, and those Christoffel symbols which are nonzero are furnished directly by the variational principle. The ones which are zero don't even appear and thus do not have to be computed explicitly. That is the gist of the "geodesic Lagrangian method" (see Homework set 3).

The geometrical meaning of the Euler-Lagrange Eq. (28.19) is quite clear: if we let the tangent to each geodesic be

$$\frac{dx^i}{ds}\frac{\partial}{\partial x^i} \equiv u^i \frac{\partial}{\partial x^i} \equiv u\,,$$

then in terms of this tangent the equation reads

$$0 = e_i\left(\frac{\partial u^i}{\partial x^k} + u^\ell \Gamma^i_{\ell k}\right)\frac{dx^k}{ds} = e_i\, u^i_{;k} u^k = \nabla_{\mathbf{u}} \mathbf{u}$$

or

$$\nabla_{\mathbf{u}} \mathbf{u} = 0 .$$

This expresses the fact that the <u>tangents of the extremal curves are autoparallel.</u>

Furthermore,

$$\frac{d}{ds}\mathbf{u}\cdot\mathbf{u} = \nabla_{\mathbf{u}}(\mathbf{u}\cdot\mathbf{u}) = 2(\nabla_{\mathbf{u}}\mathbf{u})\cdot\mathbf{u} = 0;$$

in other words, the magnitude $\mathbf{u}\cdot\mathbf{u}$ of the tangent to an extremal curve is constant along the geodesic.