# TENSOR CALCULUS 8

## 8. Covariant Derivative vs. Lie Derivative

8.1 Parallel Vector Fields

8.2 The Torsion Tensor Fields

8.3 Burger's Vector and Cartan's Torsion Tensor

Quote of the Day:

"In the long ago, no man was a man who could not wield a sword. Today nobody can be any body who cannot express himself with clarity and force."

— Anon —

## 2.7 Covariant Derivative vs Lie Derivative

Given two vector fields, say $\mathbf{u}$ and $\mathbf{v}$, the covariant derivative $\nabla_{\mathbf{u}} \mathbf{v}$ and the commutator $[\mathbf{v}, \mathbf{u}] = \mathcal{L}_{\mathbf{v}} \mathbf{u}$, also known at the "Lie derivarite with respect to $\mathbf{v}$." are two ways of generating new vectors at each point of the nonlinear manifold. One is based on a parallel transport structure, the other on the differential structure. One can, however, point to a more specific distinction between the two: the covariant derivative is a <u>tensor map</u> while the Lie bracket is not.

Question: In what sense is $\nabla_{\mathbf{u}} \mathbf{v}$ a tensor map?

Answer: Given a vector

$$\nabla \mathbf{v} : V_p \to V_p$$

$$\mathbf{u} \rightsquigarrow \nabla_{\mathbf{u}} \mathbf{v}$$

is a <u>tensor map</u> because $\nabla \mathbb{V}$ is <u>pointwise linear</u>, i.e.

$$f\mathbb{u}_1 + g\mathbb{u}_2 \rightsquigarrow f\nabla_{\mathbb{u}_1}\mathbb{V} + g\nabla_{\mathbb{u}_2}\mathbb{V}.$$

Thus $\nabla \mathbb{V}$ is a <u>tensor (map)</u> at each point $\mathcal{P}$.

By contrast, the Lie derivative (or "Lie bracket")

$$\mathcal{L}_{\mathbb{V}}: \quad V_{\mathcal{P}} \longrightarrow V_{\mathcal{P}}$$
$$\mathbb{u} \rightsquigarrow \mathcal{L}_{\mathbb{V}}\mathbb{u} \equiv [\mathbb{V}, \mathbb{u}]$$

is <u>not</u> a tensor map because

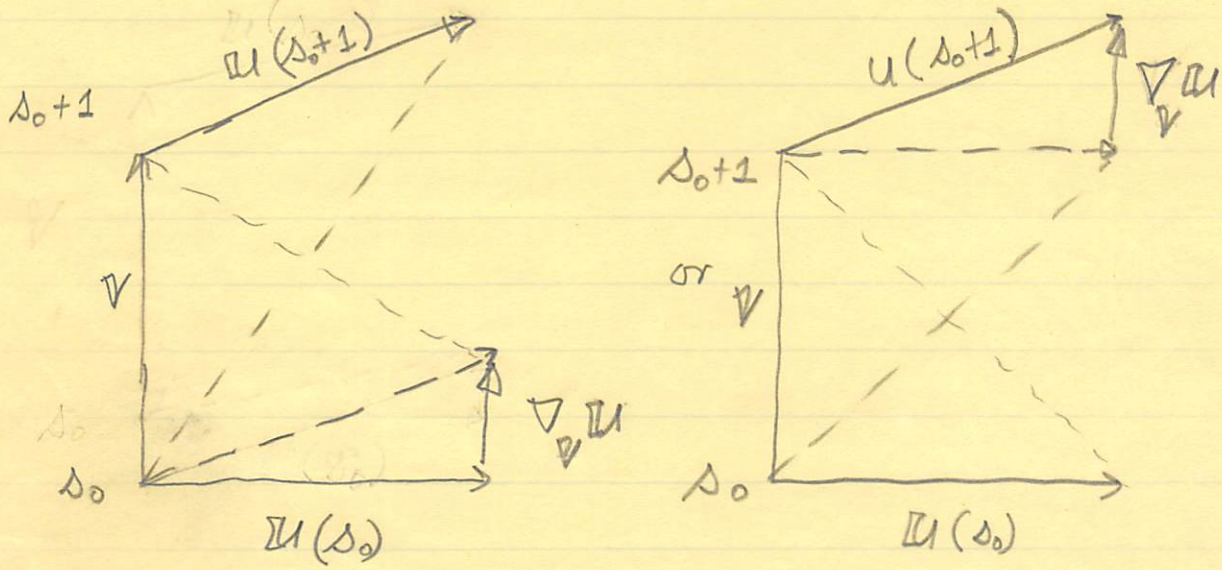
$$f\mathbb{u} \rightsquigarrow [\mathbb{V}, f\mathbb{u}] = f[\mathbb{V}, \mathbb{u}] + \mathbb{V}(f)\,\mathbb{u}$$
$$\left(= f\mathcal{L}_{\mathbb{V}}\mathbb{u} + \mathbb{u}\mathcal{L}_{\mathbb{V}}(f)\right)$$

is not pointwise linear.

## 2.8 Parallel Vector fields

Once we are given the covariant derivatives $\nabla_{e_i} e_j$, or $\Gamma^k_{ji}$, or equivalently $d e_j = e_k \otimes \omega^k_j$

then one can construct parallel vector fields. In general, a vector field $\mathbb{u}$ is not parallel along a given curve, i.e




which means, in general

$$\nabla_{\mathbb{v}} \mathbb{u} \neq 0$$

But if it is parallel then one obtains the differential equation for $\mathbb{u} = u^i e_i$

$$\nabla_{\mathbb{v}} \mathbb{u} = 0 \quad \text{or} \quad \frac{du^j}{ds} + \Gamma^j_{ik} u^i v^k = 0$$

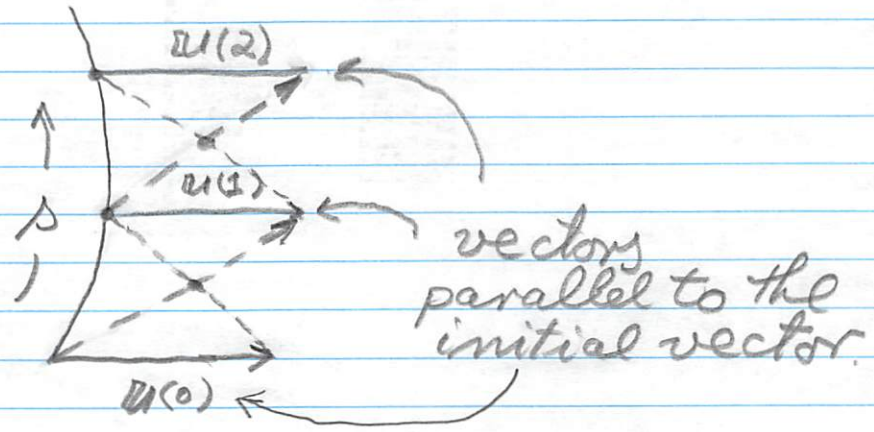
and the initial conditions are

$$u(s=s_0) = u(s_0)$$

The solution to this system of differential equations determines a parallel vector field $u(s)$ along the given curve $c^i(s)$ whose tangent is $v^i = \frac{dc^i}{ds}$

3. Torsion: Types of parallelism.

The kind of parallism exhibit by this parallel vector field is determined by the given Christoffel coefficients $\Gamma^j_{ik}$. There are two types of parallelism: those where parallel vector fields form closed parallelograms (on an infinitesimal scale) and those where the parallelograms are non closed. For example, Euclid's construction of parallel vectors along any nonstraight curve has the property that parallelograms are closed.




"Schild's Ladder"


Example: Parallel vectors on a circle of constant latitude on the 2-sphere.

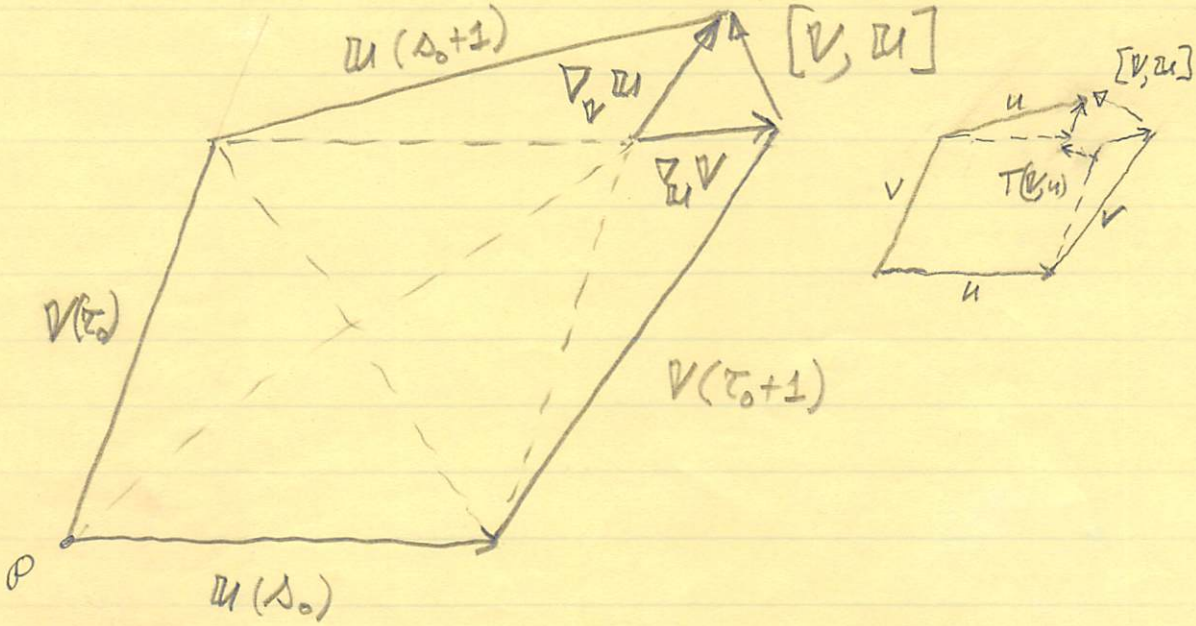
Question:
How do we characterize the geometrical nature of a given law of parallel transport?

Answer:
We construct an infinitesimal parallelogram, and see whether it's open or closed.

## 3.1 Closed Parallelogram

Infinitesimal parallelograms are constructed with the help of the covariant derivative. Suppose we are given, or we chose, two vector fields, $\mathbf{u}$ and $\mathbf{v}$. At each point the covariant derivatives $\nabla_{\mathbf{u}}\mathbf{v}$, $\nabla_{\mathbf{v}}\mathbf{u}$ and the commutator $[\mathbf{v}, \mathbf{u}]$ determine a closed parallelogram as follows:

vector fields $\mathbb{V}$ and $\mathbb{U}$ as follows




In order that the parallel transport be "consistent" it is necessary that the parallelogram at $P$ be a <u>closed</u> figure. This is expressed by the equation

$$\nabla_{\mathbb{V}}\mathbb{U} - \nabla_{\mathbb{U}}\mathbb{V} - [\mathbb{V},\mathbb{U}] \equiv \mathbb{T}(\mathbb{V},\mathbb{U}) \underset{\text{(closed triangle)}}{=} 0$$

If $\mathbb{T}(\mathbb{V},\mathbb{U}) \neq 0$ then the parallel transport is said to be <u>non-integrable</u>, or equivalently, the <u>Cartan torsion</u> at $P$ is non-zero.

$T$ is an interesting map.

It turns out that $T$ is a tensor, i.e we have the following proposition

Proposition (Cartan's Torsion Tensor)

$T$ is pointwise linear in $u$ & $v$, i.e.

$$T \quad V_P \times V_P \longrightarrow V_P.$$

$$u, v \rightsquigarrow T(v,u) = \text{above expression}$$

is such that

$$T(fv, u) = f\,T(v,u)$$

$$T(v, gu) = g\,T(v,u)$$

at each point $P$. This means that $T$ is a tensor at each point $P$.

Remark: The parallel transport is said to be <u>consistent</u> or <u>integrable</u> if this tensor vanishes identically.

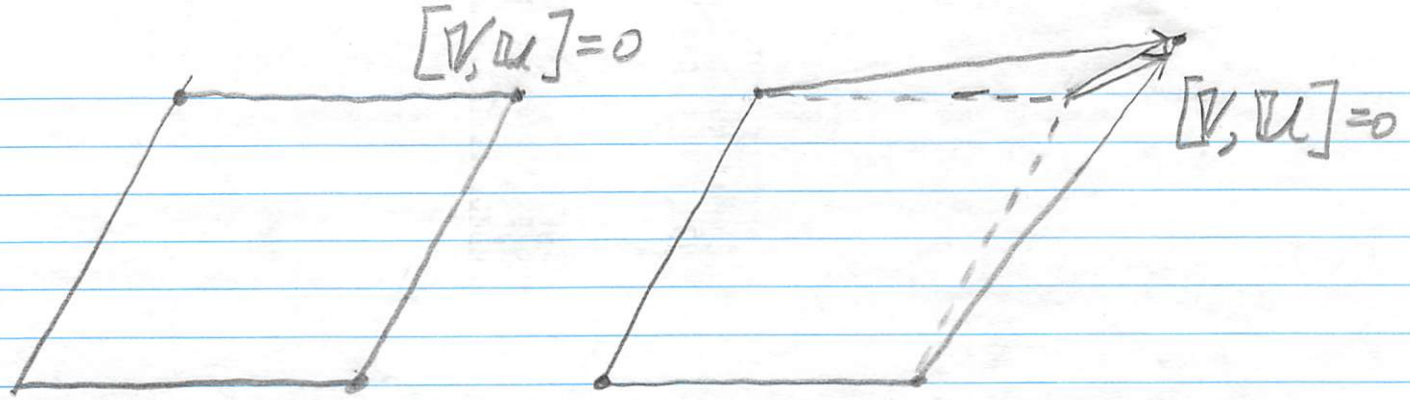
# 3.2 Burger's Vector and Cartan's Torsion Tensor.

In elasticity theory one compares the strain configuration of a continuous medium before and after the application of some stress. At each point of the medium one introduces a set of basis vectors

In the unstrained configuration the basis at one point is usually taken to be parallel to the basis at at another point.

In a strained configuration the basis vectors are however no longer parallel in general.

Let $u$ and $v$ be vectors associated with pairs of atoms of a perfect crystaline (or polycrystaline) medium




Unstrained state  Strained state

Strain induced covariant derivates:

For unstrained state: $[\mathbb{U}, \mathbb{V}] = 0$; $\nabla_{\mathbb{U}} \mathbb{V} = \nabla_{\mathbb{V}} \mathbb{U} = 0$

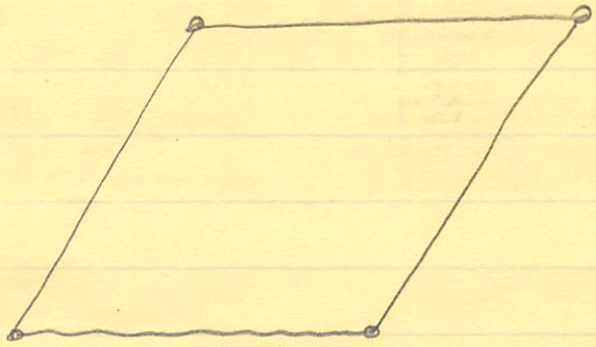
$(\Gamma^j_{ik} = 0)$

For strained state: $[\mathbb{U}, \mathbb{V}] = 0$; $\nabla_{\mathbb{U}} \mathbb{V} \neq 0$, $\nabla_{\mathbb{V}} \mathbb{U} \neq 0$

$(\Gamma^j_{ik} \neq 0)$

SKIP.

$[\mathbb{V},\mathbb{U}]=0$

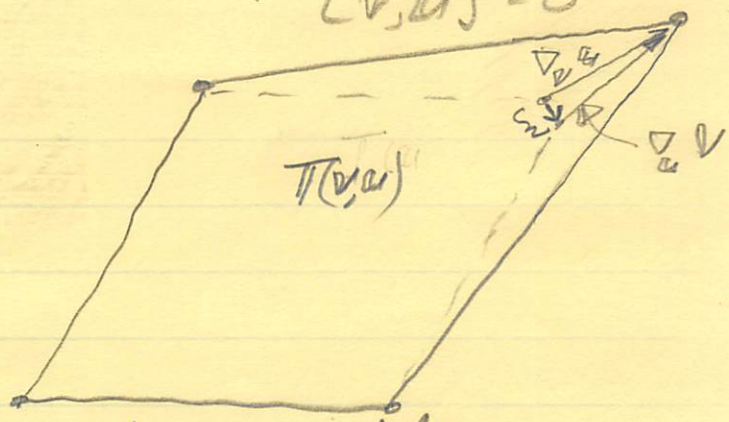


unstrained state:

$\mathbb{U},\mathbb{V}$ : $\nabla_{\mathbb{U}}\mathbb{V},\ \nabla_{\mathbb{V}}\mathbb{U}=0$

$(\Gamma^{j}_{ik}=0)$

$[\mathbb{V},\mathbb{U}]=0$




strained state

$\nabla_{\mathbb{U}}\mathbb{V},\ \nabla_{\mathbb{V}}\mathbb{U}\neq 0$

$(\Gamma^{j}_{ik}\neq 0)$

---

If the vector ("Burger's vector") obtained from the torsion tensor $\mathbb{T}$,

$$\mathbb{T}(\mathbb{V},\mathbb{U}) = \nabla_{\mathbb{V}}\mathbb{U} - \nabla_{\mathbb{U}}\mathbb{V} - [\mathbb{V},\mathbb{U}],$$


associated with the strained state vanishes then one says that dislocations, or rather the <u>dislocation density</u> is absent.

If the Burger vector

$$\mathbb{T}(\mathbb{V},\mathbb{U})\neq 0$$

then this fact expresses the existence of a non-zero dislocation density.

SK18

If $T(\mathbb{v}, \mathbb{u})$ points <u>out</u> of the plane spanned by $\mathbb{v}$ and $\mathbb{u}$ then one has a <u>density of screw dislocations</u>

If $T(\mathbb{v}, \mathbb{u})$ lies <u>in</u> the plane of $\mathbb{v}$ and $\mathbb{u}$ then one has a <u>density of edge dislocations</u>




A single "edge dislocation".

Screw dislocation

For applications to continuum mechanics see, e.g. "<u>Continuum Mechanics</u>", by W. Jaunzemis (QA 808.2 J3)
or Theory of Elasticity, by Landau & Lifschitz

It is clear that in a crystalline elastic medium the presence or the absence of imperfections ("dislocations") is quite distinct from the medium being in a strained or an unstrained state.

Suppose the medium is in an <u>unstrained</u> state. This is described by the condition

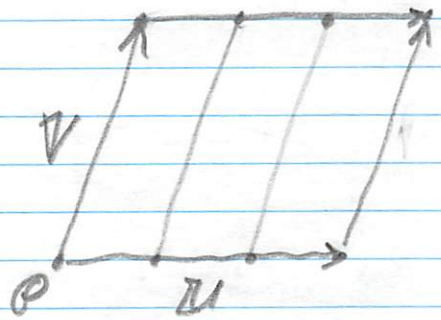
$$\nabla_{\mathbf{u}} \cdot \mathbf{v} = \nabla_{\mathbf{v}} \mathbf{u} = 0$$

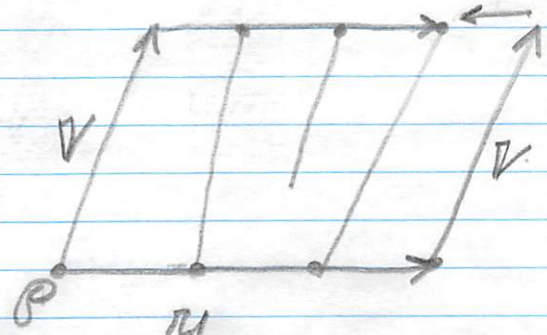
Then there are two possibilities:
(i) Dislocation are absent, and (ii) dislocations are present

Zero Strain

Zero Strain, Burger vector $\neq 0$

$T(\mathbb{u},\mathbb{v}) = [\mathbb{v},\mathbb{u}] \neq 0$




$T(\mathbb{v},\mathbb{u}) = -[\mathbb{v},\mathbb{u}]$

$T(\mathbb{v},\mathbb{u}) = \nabla_{\mathbb{v}}\mathbb{u} - \nabla_{\mathbb{u}}\mathbb{v} - [\mathbb{v},\mathbb{u}]$

$T(\mathbb{u},\mathbb{v}) = \nabla_{\mathbb{u}}\mathbb{v} - \nabla_{\mathbb{v}}\mathbb{u} - [\mathbb{u},\mathbb{v}]$

$= 0 - 0 - 0$

(Zero density of dislocations)

$T(\mathbb{u},\mathbb{v}) = \nabla_{\mathbb{u}}\mathbb{v} - \nabla_{\mathbb{v}}\mathbb{u} - [\mathbb{u},\mathbb{v}]$

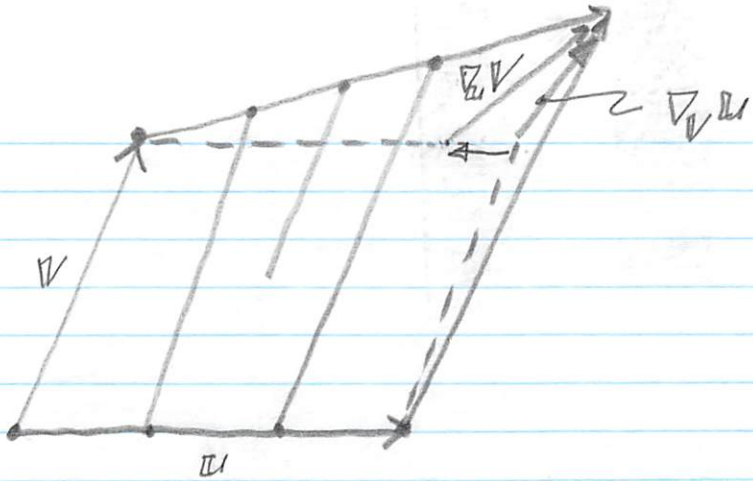
$= 0 - 0 + [\mathbb{v},\mathbb{u}]$

(Non-zero density of dislocations (skrew or edge))

Burger's vector

$T(\mathbb{u},\mathbb{v}) \neq 0$

Suppose that the medium is in a strained state, and suppose the strain is big enough so as to close the gap between the two atoms diagonally opposite the reference atom at P




$$T(\mathbb{U},\mathbb{V}) = \nabla_{\mathbb{U}}\mathbb{V} - \nabla_{\mathbb{V}}\mathbb{U} - [\mathbb{U},\mathbb{V}]$$
$$= \nabla_{\mathbb{U}}\mathbb{V} - \nabla_{\mathbb{V}}\mathbb{U} - 0$$
$$\neq 0$$

(Non-zero strain and non-zero density of dislocations)

In general, Burger's vector is defined in terms of <u>Cartan's torsion tensor</u>:

$$T: \quad \mathbb{U},\mathbb{V} \rightsquigarrow T(\mathbb{U},\mathbb{V}) = \nabla_{\mathbb{U}}\mathbb{V} - \nabla_{\mathbb{V}}\mathbb{U} - [\mathbb{U},\mathbb{V}]$$

= amount of dislocation per unit cell

= Burger's vector