TENSOR CACLULUS 9

9. Curvature

9.1 Rotation of a Vector (Parallel) Transported Around a Loop

9.2 Riemann Curvature Tensor

not developed in class $\left\{ \begin{array}{l} \text{9.3 Equation of Geodesic Deviation} \\ \text{9.4 Rotation = Relative Acceleration} \end{array} \right.$

Wheeler's First Moral Principle

Never make a calculation until you know the answer. Make an estimate before every calculation, try a simple physical argument (symmetry! invariance! conservation!) before every derivation, guess the answer to every puzzle.

(Taken from Space-Time Physics, 2nd Edition, Taylor & Wheeler 1993)

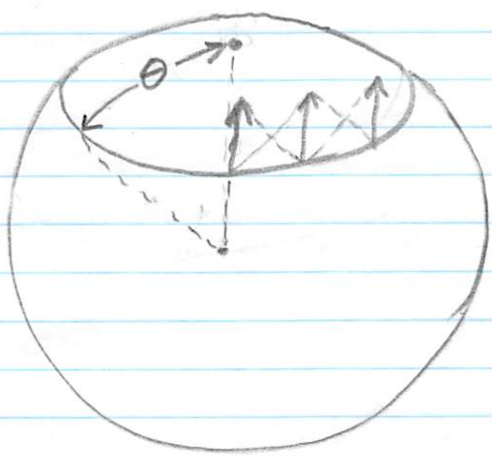
Curvature

A given law of parallel transport provides the rules for constructing parall vectors. The application of this construction to two vectors emanating from the same point yields a parallelogram. However, whether or not the parallelogram is a closed figure depends on the torsion (tensor) of the parallel transport law. The displacement necessary to form a closed figure is expressed by the (vectorial) value of this torsion tensor applied to the two vectors spanning the parallelogram. Thus parallel transport has a displacement aspect to it, and torsion, via parallelograms reveals this displacement

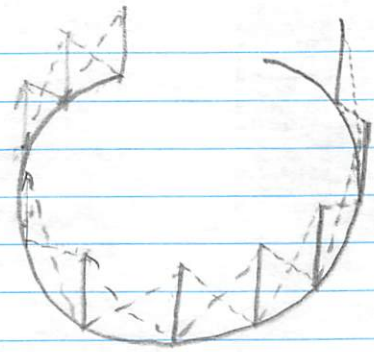
But *parallel transport also has a rotation aspect to it*, and it is revealed by means of the curvature via vectors parallel-transported around a closed curve.

4.1 Rotation of a Vector Transported Around a Closed Loop.

Let us illustrate this rotation on the parallel transport characterized by Euclid's closed parallelogram construction on the two sphere S^2.



We take a closed circle of constant latitude θ, and using the closed parallelogram construction, parallel transport a vector around this closed loop.

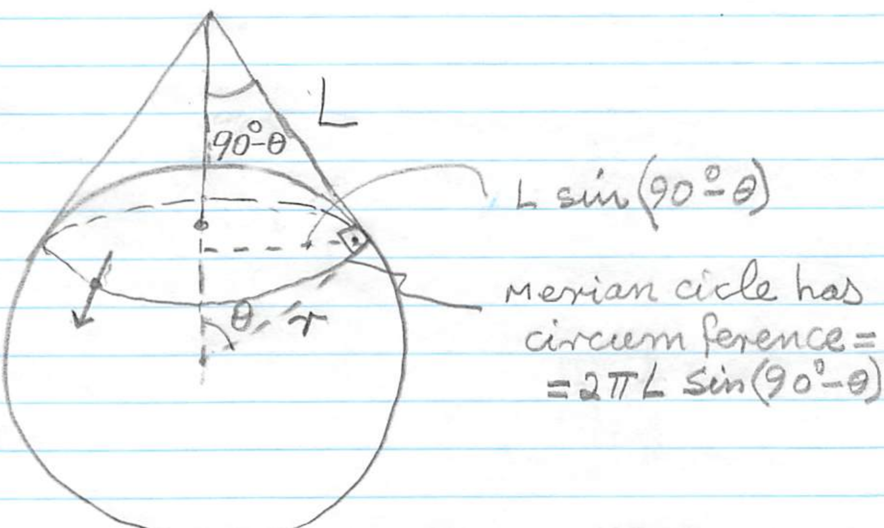


The vectors are parallel because they are the opposite sides of a succession of parallelograms.

It is evident that the successive opposing sides of these parallelograms receive their parallelness from the familiar parallel transport of the ambient three dimensional space. In other words, the parallelism of Euclidean three dimensional space induces a unique parallel transport on the two sphere S^2.

Q: What is the result of this transport?

A: The result of this parallel transport can be exhibited quantitatively by cutting out an annular strip surrounding the circle of constant latitude, θ=constant. This annular strip is tangent to the bottom of a cone whose apex angle is $2 \times (90^\circ - \theta)$.



Cut this cone along an edge, and flatten the cone out so that it becomes a disk with a missing sector of angle α.

The parallel vector field is along the arc perimeter of the disk. It is evident that after a full circuit of parallel transport, a vector gets rotated by the angle α.

More precisely, we have

α = Rotation angle of a vector after parallel transport around a meridean circle of latitude θ away from the North pole

$$= \frac{\text{arclength of completed circle} - \text{arclength of meridean circle}}{L}$$

$$= \frac{2\pi L - 2\pi L \sin(90^\circ - \theta)}{L}$$

$$= 2\pi(1-\cos\theta)$$

This angle leads us to the following tentative definition:
The <u>curvature</u> permeating an <u>enclosed area</u> yields the angular rotation suffered by a vector parallel transported around the perimeter of that area:

$$\boxed{\alpha \equiv (\text{"curvature"}) \times \text{enclosed area}}$$

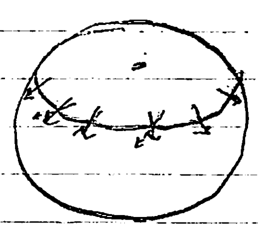
intrinsic to S^2

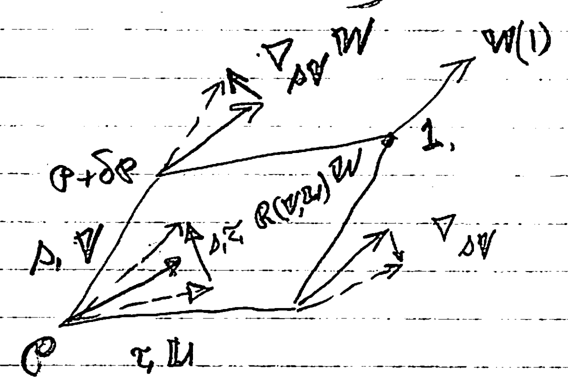
or

$$\text{curvature} \equiv \frac{\alpha}{\underbrace{\int_0^{2\pi} d\varphi \int_0^{\theta} r^2 \sin\theta \, d\theta}_{\text{enclosed area}}} = \frac{2\pi(1-\cos\theta)}{2\pi r^2(1-\cos\theta)} = \frac{1}{r^2}$$

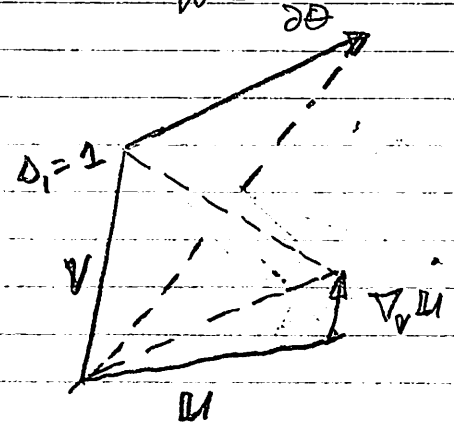
The radius r is called the <u>radius of curvature</u> of the enclosed area.

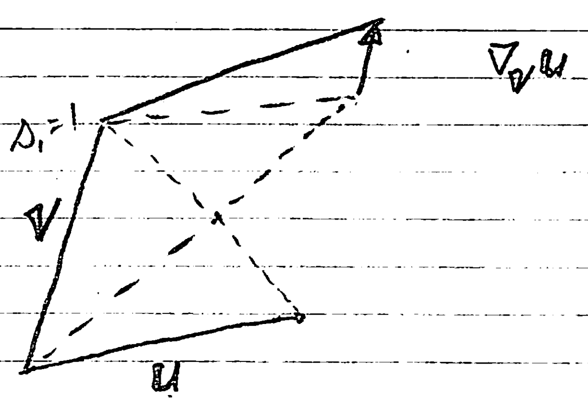
We shall now generalize this definition of curvature by having the enclosing not be a smooth circle, but by a polygon whose perimeter is determined by the intersecting curves tangent to a chosen pair of vector fields ($\mathbb{U}$ and $\mathbb{V}$). Furthermore, the non-parallel fiducial vector field $\frac{\partial}{\partial\theta}$, which makes the parallel transport visible, we shall call $\mathbb{W}$





$\mathbb{W} = \frac{\partial}{\partial\theta}$





Definition: $\mathbf{W}$ is a *parallel* vector field along the curve through $\mathbf{V} \Longleftrightarrow$

$$\nabla_{\mathbf{V}} \mathbf{W} = 0$$

Definition: A curve is called a *geodesic* if its tangent vector is parallel to itself:

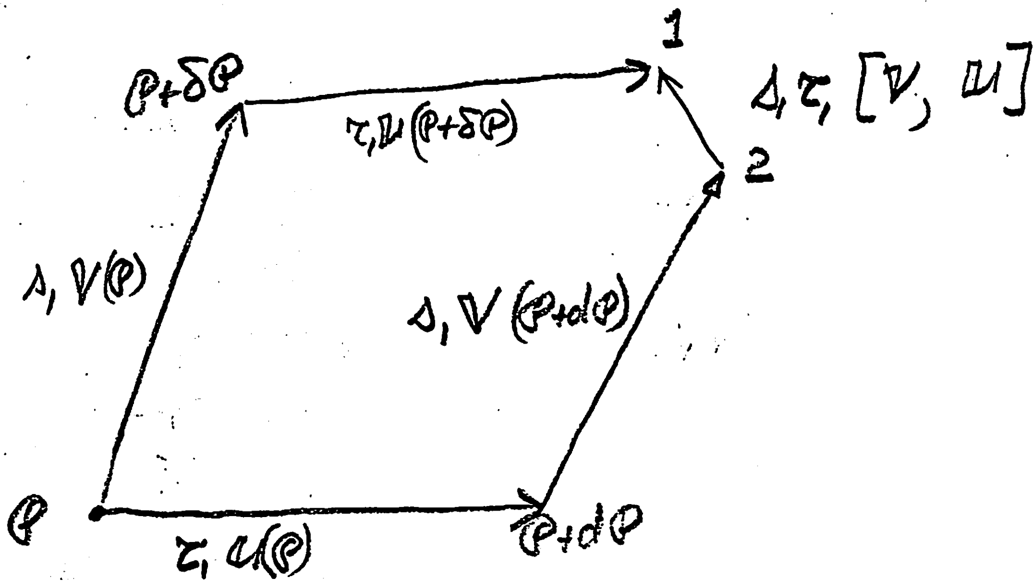
$$\boxed{\nabla_{\mathbf{V}} \mathbf{V} = 0}$$

(This is helpful for problem 10.16)

11.4 in MTW

II Curvature

a) Consider a closed curve determined by the vector fields $\mathbf{U}$ and $\mathbf{V}$



b) (i) On this closed curve introduce an arbitrarily specified "fiducial" vector field $\mathbf{W}$

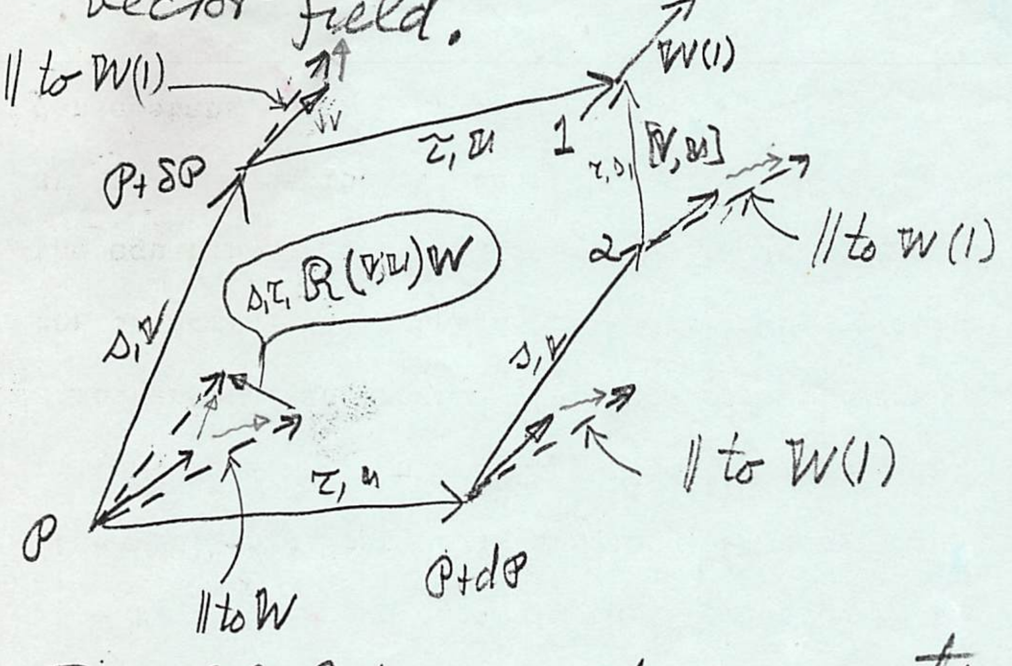
(ii) We wish to parallel transport $\mathbf{W}(1)$ to point $\mathcal{P}$

$1 \to \mathcal{P}+\delta\mathcal{P} \to \mathcal{P}$ along 1st broken path

$1 \to 2 \to \mathcal{P}+d\mathcal{P} \to \mathcal{P}$ along 2nd broken path

(iii) (The purpose of the vector field $\mathbb{W}$) is to make this parallel transport visible and documentable along each leg (individually) of the two broken paths

(iv) Even though one uses this intermediate fiducial vector field $\mathbb{W}$ along the two broken paths, ~~never the less~~, once the parallel transport of $\mathbb{W}(1)$ from point 1 to point P has been performed along the two paths, the difference between the different results will be independent of the ~~the~~ intermediate vector field.



solid vectors are part of the fiducial vector field $\mathbb{W}$

Dashed vectors are vectors parallel to $\mathbb{W}(1)$ along the obvious paths

c) Parallel transport along 1st broken path:

(1) $\mathbb{W}(P+\delta P) + \tau_1 \nabla_u \mathbb{W}\Big|_{P+\delta P} + \frac{\tau_1^2}{2}\nabla_u\nabla_u \mathbb{W}\Big|_{P+\delta P} + \cdots$ = vector at $P+\delta P$ which

is parallel to $W(1)$. This vector has been parallel translated from 1 to $P+\delta P$.

② The parallel translate to point P of the vector in ① is

$$\underbrace{W(P) + s_1 \nabla_V W\Big|_P + \frac{s_1^2}{2} \nabla_V \nabla_V W}_{\text{Parallel translate of } W(P+\delta P)} +$$

$$+ \underbrace{\tau_1 \nabla_U W\Big|_P + s_1\tau_1 \nabla_V \nabla_U W\Big|_P + \cdots}_{\parallel \text{ translate of } 2^{nd} \text{ term}}$$

vector $\parallel$ to a sum = sum of $\parallel$ vectors

$$+ \frac{\tau_1^2}{2} \nabla_U \nabla_U W\Big|_P =$$ vector at P which is $\parallel$ to $W(1)$. This vector has been parallel translated $1 \to P+\delta P \to P$.

d) Parallel transport along 2^{nd} broken path

① $W(1) \to W(2) + s_1\tau_1 \nabla_{[V,U]} W\Big|_2 = \parallel$ translate of $W(1)$ at point 2.

② $W(P+dP) + s_1 \nabla_V W\Big|_{P+dP} + \frac{s_1^2}{2} \nabla_V \nabla_V W\Big|_{P+dP}$

$+ s_1\tau_1 \nabla_{[V,U]} W\Big|_{P+dP} = \parallel$ translate of above to $P+dP$.

Step

③ (same as c) ② except $\tau, U \leftrightarrow s, V$)

$+ s_1\tau_1 \nabla_{[V,U]} W = \parallel$ translate of above to P.

e) (Parallel translate of $W(1)$ to point P around the

1^{st} broken path) − (parallel translate of $\mathbb{W}(1)$ to point $\mathcal{P}$ around the 2^{nd} broken path)

$=$ (vector in c②) − (vector in d③) $=$

$$\boxed{\begin{aligned} &= s_1 \tau_1 \left[\nabla_{\mathbb{V}} \nabla_{\mathbb{U}} - \nabla_{\mathbb{U}} \nabla_{\mathbb{V}} - \nabla_{[\mathbb{V},\mathbb{U}]}\right] \mathbb{W} = s_1 \tau_1 \mathcal{R}(\mathbb{V},\mathbb{U}) \mathbb{W} \\ &= s_1 \tau_1 \text{ "Riemann}(\dots, \mathbb{W}, \mathbb{V}, \mathbb{U})\text{"} = \text{a vector at } \mathcal{P}. \\ &= \text{vectorial amount of rotation that } \mathbb{W}(\mathcal{P}) \text{ undergoes by parallel translating it around the closed quadrilateral.} \end{aligned}}$$

<u>$\mathcal{R}$ is point wise linear</u>:

$$\mathcal{R}(f\mathbb{V}, \mathbb{U})\mathbb{W} = f\,\mathcal{R}(\mathbb{V},\mathbb{U})\mathbb{W}$$
$$\mathcal{R}(\mathbb{V}, g\mathbb{U})\mathbb{W} = g\,\mathcal{R}(\mathbb{V},\mathbb{U})\mathbb{W}$$
$$\mathcal{R}(\mathbb{V}, \mathbb{U})h\mathbb{W} = h\,\mathcal{R}(\mathbb{V},\mathbb{U})\mathbb{W}$$

where f, g and h are functions.

Conclusion: $\boxed{\mathcal{R} \text{ is a tensor}}$