CALCULUS OF VARIATIONS
and
TENSOR CALCULUS

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Beta Edition

Contents

Chapter 1

FUNDAMENTAL IDEAS

Lecture 1

1.1 Multivariable Calculus as a Prelude to the Calculus of Variations.

Calculus of several variables deals with the behaviour of (multiply) differentiable functions whose domain is spanned by a finite number of coordinates.

The utility of this calculus stems from the fact that it provides, among others, methods for finding the "critical" points of a given function. For, say $f(x^1, x^2, \cdots, x^n)$, this function has points where it has an extremum, which is characterized by

$$\frac{\partial f}{\partial x^i}(x_0^1, x_0^2, \ldots, x_0^n) = 0$$

The critical point $\vec{x}_0 = (x_0^1, x_0^2, \ldots, x_0^n)$ is a point where the function has a maximum, a minimum or an inflection behaviour. Which type of these extremal behaviours the function has in the neighborhood of its critical point can, in general, be inferred from the properties of the (symmetric) second derivative matrix,

$$\left[\frac{\partial^2 f(\vec{x}_0)}{\partial x^i \partial x^j}\right] \quad \left.\begin{matrix} i \\ j \end{matrix}\right\} 1, \ldots n$$

at the critical point $\vec{x}_0$.

The *calculus of variations* extends these methods from a finite to an infinite dimensional setting.

The distinguishing feature is that each point in this infinite dimensional setting is a *function*. Furthermore, the task of finding the extremum of a function becomes the task of finding the extremum of a "functional".

A "functional" is a map whose domain is the infinite dimensional space, each of whose points is a function. A "critical" point in this space is an "optimal" function which maximizes, minimizes, or in general, extremizes the given functional.

1.2 Some Typical Problems in the Calculus of Variations.

Nearly all problems in mathematical engineering, physics and geometry are closely related in one form or another to the calculus of variations. It turns out that these problems can be expressed in terms of some sort of optimization principle which says that some functional must be maximized, minimized or, in general, extremized. Let us consider some typical examples of such problems.

Problem 1. *The Brachistochrone Problem.*

Given: A particle moves along a curve $y(x)$ without friction. The particle starts with zero velocity from point $(x_1, y_1) = (0, 0)$ and advances, because of a vertical constant gravitational field, to the point (x_2, y_2).

Question: For what curve is the travel time a minimum?

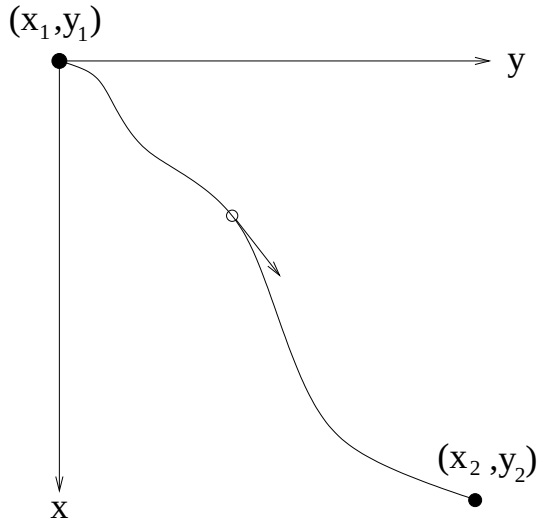


Figure 1.1: Particle sliding down a bent wire.

Let us formulate the problem more precisely. The travel time is given by the integral

$$\text{time} = \int_{t_1}^{t_2} dt = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{d(\text{path length})}{\text{speed}} \equiv \int_1^2 \frac{ds}{v} \ .$$

For a typical curve $y(x)$ with slope $\dfrac{dy}{dx}$, the speed v and the path length differentials are obtained from the following considerations:

(a) *time independent* gravitational field *implies* that the *total energy* of a system *is conserved*, i.e. independent of time. Hence,

$$\text{T.E.} \equiv \text{K.E.}+\text{P.E.} = \text{constant} \ ,$$

or

$$\frac{1}{2}mv^2 - mgx = 0 \ .$$

Here we have chosen the P.E. energy to be zero at $x = 0$:

$$\text{P.E.}(x = 0) = 0$$

Thus

$$v = \sqrt{2gx} \ .$$

Comment: The fact that the total energy is independent of time,

$$\text{T.E.}=\text{constant} \ ,$$

is an illustration of a general principle, the principle of energy conservation. It applies to all dynamical systems evolving in a time-invariant environment, e.g. a freely swinging pendulum or vibrating body. It does not apply to dynamical systems subjected to externally applied driving forces which depend on time.

(b) The element of path length is

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \ .$$

The total travel time is therefore

$$t_{12} = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{2gx}} \ dx \ ,$$

which is a function of the function $y(x)$, i.e. t_{12} is a “functional".

Thus, to solve the brachistochron problem requires that we find a function $y(x)$ such that the "functional" t_{12}=minimum!

Problem 2. *Minimum Surface of Revolution.*

Given: Some curve between two fixed points (x_1, y_1) and (x_2, y_2) in the plane $z = 0$. A surface of revolution is formed by revolving the curve around the y-axis.

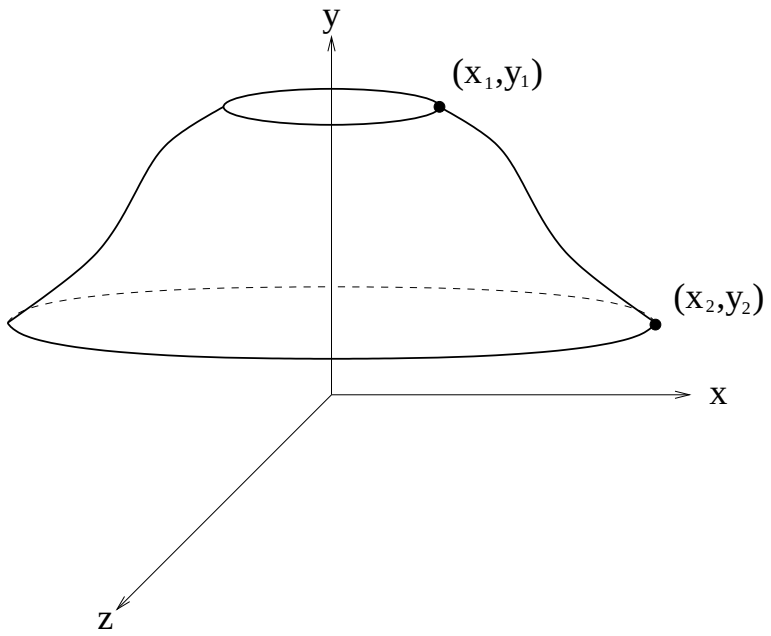


Figure 1.2: Surface area of revolution generated by a curve in the $x - y$ plane.

Question: For what curve $y(x)$ is the surface area a minimum?

One must find a function $y(x)$ such that the functional

$$\begin{aligned} \text{Area}[y] &= \int_1^2 2\pi x ds = 2\pi \int_{(x_1,y_1)}^{(x_2,y_2)} x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \text{minimum!} \end{aligned}$$

Problem 3. *Fermat's Principle of Least Time.*

Given: An inhomogeneous medium through which light (or sound) can propagate with a velocity that depends on position,

$$v(x, y) = \frac{c}{n(x, y)} \ .$$

Here $n(x, y)$ is the position dependent refractive index, which is assumed to be given.

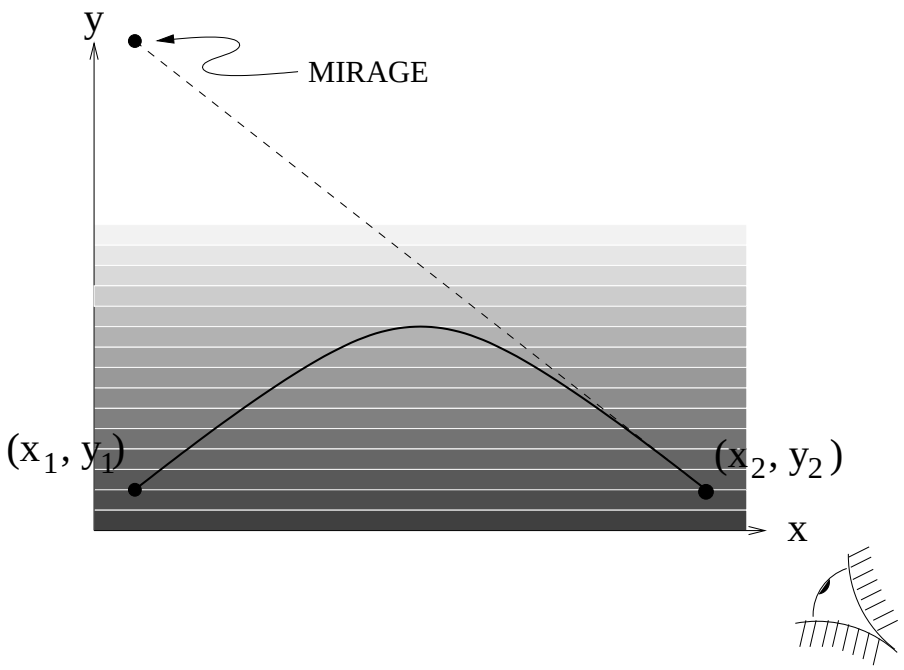


Figure 1.3: Optical path as determined by Fermat's principle of least time. In the lighter regions the propagation speed of light is faster. Thus the optimal (least time) path includes such regions in its domain of passage.

Statement of Fermat's Principle:
A light (or a radio, or a sound) beam takes that path $y(x)$ through two points (x_1, y_1) and (x_2, y_2) for which the travel time is a minimum.

This principle subsumes many natural phenomena, including 1.) mirages [1], 2.) ionospheric bending of radio beacons, 3.) twinkling of stars[2], 4.) sonar beams refracted by underwater temperature and salinity changes, 5.) light beam traversing an optical interface, etc.

The application of Fermat's principle consists of finding $y(x)$ such that

$$\begin{aligned} t_{12} &= \int_1^2 \frac{ds}{v(x,y)} \\ &= \int_{x_1}^{x_2} n(x,y)\frac{\sqrt{1+(y')^2}}{c}\,dx = \text{minimum!} \end{aligned}$$

[1]Due to a refractive index increasing with altitude.
[2]Due to light rays from a star propagating through atmospheric plumes of warm and cold air.

1.3 Methods for Solving Problems in Calculus of Variations.

There are three known ways of finding an optimal function which minimizes (or maximizes) the given integral expression for the functional.

1. the method of finite differences,
2. the method of variations,
3. the direct method via minimizing sequences.

We shall briefly discuss the first, extensively develop the second, and postpone discussion of the third until later.

1.3.1 Method of Finite Differences.

In order to appreciate the basic meaning of problems in the calculus of variations and methods for solving them, it is important to see how they are related to the calculus of n variables. We are given a functional of the form

$$J[y] = \int_a^b F(x, y, y')dx \quad \text{with } y(a) = A \text{ and } y(b) = B.$$

This is a map which assigns a number to each curve $y(x)$ passing through the two points (a, A) and (b, B). One may find the curve which minimizes the integral J by the following approximation scheme, which is based on the construction of a Riemann sum with the derivative function approximated by the partition-induced set of differential quotients:

1. Subdivide the interval $[a, b]$ into $n + 1$ equal parts by using the points

$$x_0 = a,\ x_1, \ldots, x_n,\ x_{n+1} = b$$

2. Replace the curve $y(x)$ with the polygonal line whose vertices are

$$(x_0, A), (x_1, y(x_1)), \ldots, (x_n, y(x_n)), (x_{n+1}, B)$$

3. Approximate the integral $J[y]$ by a finite Riemann sum,

$$J(y_1, \ldots, y_n) = \sum_{i=1}^{n+1} F\left(x_i, y_i, \frac{y_i - y_{i-1}}{\Delta x}\right)\ \Delta x\ ,$$

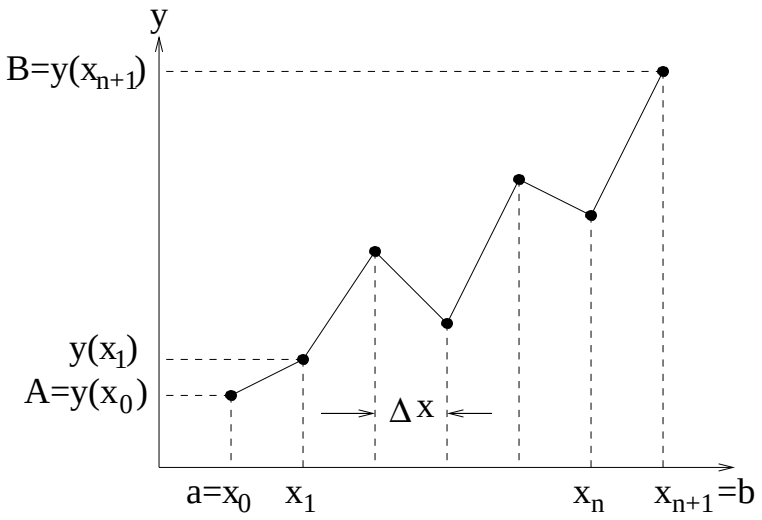


Figure 1.4: Curve approximated by a polygonal line.

where

$$y_i = y(x_i) \text{ and } \Delta x = x_i - x_{i-1}$$

Thus, each polygonal line is uniquely determined by the ordinates $y_1, y_2, \ldots, y_n$ of the vertices because the end points are assumed to be *fixed*. Consequently, the *functional* $J[y]$ may be considered as *being approximated* by the function $J(y_1, \ldots, y_n)$, and the problem of finding the optimal curve $y(x)$ which minimizes $J[y]$ may be approximated by the problem of finding the critical point $(y_1^*, \ldots, y_n^*)$ which minimizes the finite Riemann sum $J(y_1, \ldots, y_n)$.

This minimization process consists of solving the following n simultaneous equations

$$\frac{\partial J}{\partial y_j} = 0 \qquad j = 1, \ldots, n$$

Using the expression for J, namely

$$\begin{aligned} J(y_1, \ldots, y_n) &= F\left(x_1, y_1, \frac{y_1 - y_0}{\Delta x}\right)\Delta x + F\left(x_2, y_2, \frac{y_2 - y_1}{\Delta x}\right)\Delta x \\ \ldots &+ F\left(x_j, y_j, \frac{y_j - y_{j-1}}{\Delta x}\right)\Delta x + F\left(x_{j+1}, y_{j+1}, \frac{y_{j+1} - y_j}{\Delta x}\right)\Delta x \\ + \ \ldots &+ F\left(b, B, \frac{B - y_n}{\Delta x}\right)\Delta x, \end{aligned}$$

let us calculate the partial derivative w.r.t y_j:

$$\begin{aligned}\frac{\partial J}{\partial y_j} = \frac{\partial}{\partial y_j}\ \{F(x_j, y_j, \underbrace{\frac{y_j - y_{j-1}}{\Delta x}}_{y'_{j-1}})\ \Delta x \quad &+ \quad F(x_{j+1}, y_{j+1} \underbrace{\frac{y_{j+1} - y_j}{\Delta x}}_{y'_j})\ \Delta x\} \\ = F_{y_j}(x_j, y_j, \frac{y_j - y_{j-1}}{\Delta x})\ \Delta x \quad &+ \quad F_{y'_j}(x_j, y_j, y'_{j-1}) \frac{1}{\Delta x} \Delta x \\ &+ \quad F_{y'_j}(x_{j+1}, y_{j+1}, y'_j)\ \frac{-1}{\Delta x}\ \Delta x\end{aligned}$$

As $\Delta x \to 0$; x_j, $x_{j-1} \to x$; $y_j \to y$ the number of subintervals increases without limit and the expression approaches zero. But if we divide by Δx, then we have

$$\lim_{\Delta x \to 0}\ \frac{1}{\Delta x}\ \frac{\partial J}{\partial y_j} \equiv \frac{\delta J}{\delta y}$$

Remark: The expression $\Delta x \partial y_j$ has a direct geometrical significance as the area between the given curve and the varied (dashed) curve. The limit of the differential

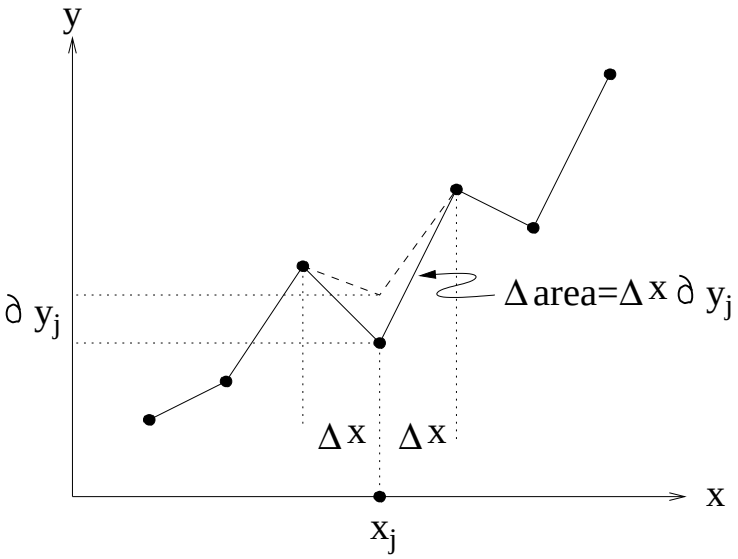


Figure 1.5: The area between the solid and the dashed curve is $\Delta x \partial y_j$. In the continuum limit, Figure 1.11, the solid (given) curve becomes a "trial function" and the dashed (varied) curve becomes its "variant".

quotient is the variational derivative of J,

$$\frac{\delta J}{\delta y} = \frac{\partial F(x, y, y')}{\partial y} - \frac{d}{dx}\ \frac{\partial F(x, y, y')}{\partial y'}$$

The process of minimizing the functional $J[y]$ by first approximating it as $J(y_1, \ldots, y_n)$, a function of the n independent variables $y_1, \ldots, y_n$, is called the *method of finite differences*.

Leonard Euler used it to solve problems in the calculus of variations. By replacing smooth curves with polygonal lines he solved such problems as multivariable problems in n-dimensions. By letting $n \to \infty$, he obtained the mathematically exact[3] solution.

Thus the functional $J[y]$ may be considered a function of infinitely many variables, and the *calculus of variations may be considered an extension* of multivariable calculus *into a space of infinite dimensions*.

Lecture 2

1.4 The Method of Variations.

The method of finite differences first approximates an infinite dimensional problem by one of finite dimensions. Solving the finite dimensional problem consists of finding the critical point of a scalar function having a finite number independent variables. The coordinates of this critical point, say $\{y_i = y(x_i) : i = 1, \ldots, n\}$, are the vertices of a polygonal curve, the finite approximation to the curve that optimizes the scalar functional. One obtains the exact solution, the optimal curve, to the infinite dimensional problem by letting n, the dimension of the approximation space, become infinite. There exist numerical optimization techniques for finding the critical point scalar function on a finite dimensional approximation space. One obtains the exact solution to the infinite problem by letting n, the dimension of the approximation space, become infinite. But in that case the numerical optimization technique becomes very computationally expensive.

However, one can dispense with the intermediate approximation process entirely by dealing with the infinite dimensional problem directly. In this strategy one does not obtain the optimal curve directly. Instead, one obtains a differential equation which this curve must satisfy. However, finding solutions to a differential equation is a highly developed science. Hence, dispensing with the intermediate approximation process is an enormous advantage if, as is usually the case, one can solve the differential equation.

It is difficult to overstate the importance of the method which allows us to do this. It is called the method of variations, which we shall now describe.

[3]Mathematical exactness is the limit of *physical exactness*. The latter presupposes a specific context of the measurements underlying the results. This context is limited both by the *range* and the *precision* of the measured data it subsumes. Thus, given such a context, a result is said to be physically exact in relation to such a specified context. The process of going to the above limit of physical exactness is made mathematically precise by means of the familiar $\delta - \epsilon$ process.

1.4.1 Variants and Variations

The basic problem of the calculus of variation is this:

Given: (a) $F(x; q, p)$ a function of three variables and (b) a pair of points (x_1, y_1) and (x_2, y_2).

Determine: That "optimal" function $y(x)$ which passes through (x_1, y_1) and (x_2, y_2) such that

$$J[y] = \int_{x_1}^{x_2} F(x; y(x), y'(x))dx \tag{1.1}$$

has the largest (or smallest or, in general, extremal) value for $y(x)$ as compared to any *variant* of $y(x)$ also passing through (x_1, y_1) and (x_2, y_2).
Remark 1: The dictionary definition of "variant" is as follows:

variant – something that differs in form only slightly from something else. Remark

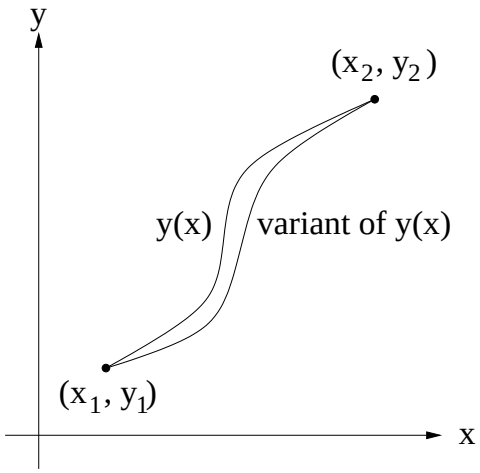


Figure 1.6: A function $y(x)$ and one of it variants

2: In order to determine whether $y(x)$, which passes through (x_1, y_1) and (x_2, y_2), is an "optimal" curve, one must compare the functional J evaluated for all variants of $y(x)$, including $y(x)$ itself.
Remark 3: We shall *assume* that an optimal curve exists, and then deduce as a consequence the conditions that $y(x)$ must satisfy. In other words, we shall determine the *necessary conditions* for $y(x)$ to exist. We are *not* developing *sufficient conditions*[4] for the existence of an optimal curve.

[4]Sufficient conditions for the functional $J[y]$ to have an extremum are discussed and given in Chapters 5 and 6 in *Calculus of Variations* by I.M. Gelfand and S.V. Fomin; Dover Publications, Inc., Mineola, N.Y., 2000.

To test whether a curve is optimal by comparing values of J, pick a curve suspected of being an optimal one and deform it slightly. This yields a *variant*, a *deformed curve*, of the original curve, namely

$$\begin{aligned} y &= f(x, \alpha_i) \\ &\equiv f(x, 0) + \alpha_1 h_1(x) + \alpha_2 h_2(x) + \dots \\ &\equiv f(x, 0) + \delta f(x) \end{aligned}$$

Here

$$y = f(x, 0) \equiv f(x)$$

is the *optimal curve* under consideration, while

$$\delta f(x) = \alpha_1 h_1(x) + \alpha_2 h_2(x) + \dots$$

is a *variation* in the function f, a variation which is a generic sum of linearly independent functions $\{h_k(x)\}$.

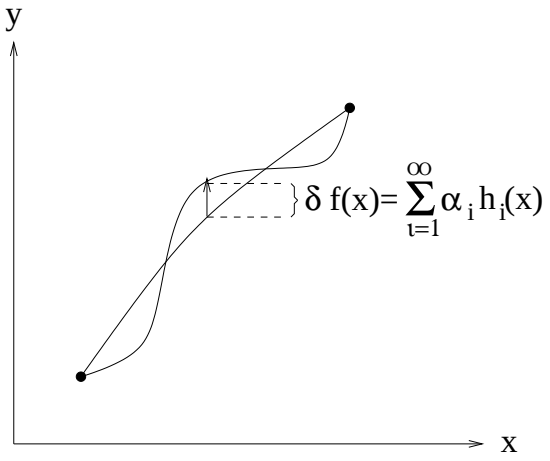


Figure 1.7: Arbitrary variation of f as a superposition of basis functions.

Definition. The fact that all variants of f pass through the same pair of endpoints we express by saying that they are *admissible* functions; in other words admissible functions have the property that their variations vanish at the endpoints,

$$\boxed{\delta f(x_1) = \delta f(x_2) = 0}.$$

Because of the linear independence of the h_i's, this is equivalent to

$$\boxed{h_i(x_1) = h_i(x_2) = 0 \quad i = 1, 2 \dots}$$

Relative to the set of basis functions $\{h_i\}$ the variants

$$\begin{aligned} A: &\quad f(x,0) - .2h_1(x) \\ B: &\quad f(x,0) - .1h_1(x) \\ C: &\quad f(x,0) \\ D: &\quad f(x,0) + .1h_1(x) \\ E: &\quad f(x,0) + .2h_1(x) \\ F: &\quad f(x,0) + .2h_1(x) + .1h_2(x) \\ G: &\quad f(x,0) + .2h_1(x) + .2h_2(x) \end{aligned}$$

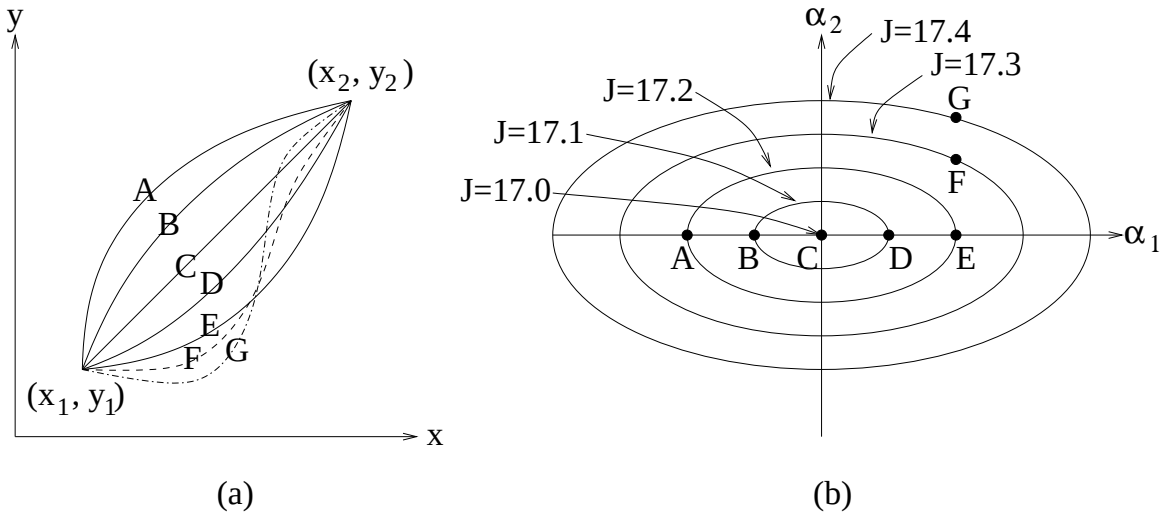


Figure 1.8: (a) Optimal curve (C) and some of its variants in x-y space. (b) Isograms of J on the function space of variants.

are represented as points

$$\begin{aligned} A: &\quad (-.2, 0, \dots) \\ B: &\quad (-.1, 0, \dots) \\ C: &\quad (0, 0, \dots) \\ D: &\quad (.1, 0, \dots) \\ E: &\quad (.2, 0, \dots) \\ F: &\quad (.2, .1, \dots) \\ G: &\quad (.2, .2, \dots) \end{aligned}$$

in the function space*coordinatized* by α_1, α_2, etc. as depicted in Figure 1.8.

Lecture 3

1.4.2 The Euler-Lagrange Equation

The problem of variational calculus is the task of optimizing Eq.(1.1). In particular, the mathematization of the calculus of variations consists of setting up the problem, defining the framework, its key concepts, and the causal relations between them.

The solution to the problem consists of identifying the optimization method in mathematical terms.

I. The Problem

The achievement of these goals requires three steps.
(i) Specify the space of trial functions on which that integral is defined,
(ii) recognize that this integral J is a mapping from these trial function into the reals, and
(iii) find the critical point(s) in the domain space for which the mapping has extremal values in this space, i.e. find the optimal function(s).

The space of trial functions is a non-linear submanifold of the linear space of functions which have the *norm*

$$\|y(x)\| = \max_{x_1 \le x \le x_2} |y(x)| + \max_{x_1 \le x \le x_2} |y'(x)| < \infty \ . \tag{1.2}$$

These functions form a vector space. Because of its norm structure it is called a "Banach" space.

The space of trial functions does *not* consist of *all* Banach space elements, only of those functions $y(x)$ whose graphs run through the fixed ends points (x_1, y_1) and (x_2, y_2) as in Figures 1.6, 1.7, 1.8(a):

$$y(x_1) = y_1 \text{ and } y(x_2) = y_2 \ . \tag{1.3}$$

The set of functions that satisfy Eqs.(1.2) and (1.3) (integrability and the end point conditions) forms the set of the already mentioned *trial functions*. For emphasis one also calls them *admissible* trial functions. Taking into account

1. that they make up the above-mentioned non-linear submanifold, S,

$$S = \{y : y(x) \text{ satisfies } Eqs.(1.2) \text{ and } (1.3)\}, \tag{1.4}$$

   the set of admissible functions, and

2. that the variational integral

$$J[y] = \int_{x_1}^{x_2} F\left(x; y, \frac{dy}{dx}\right) dx. \tag{1.5}$$

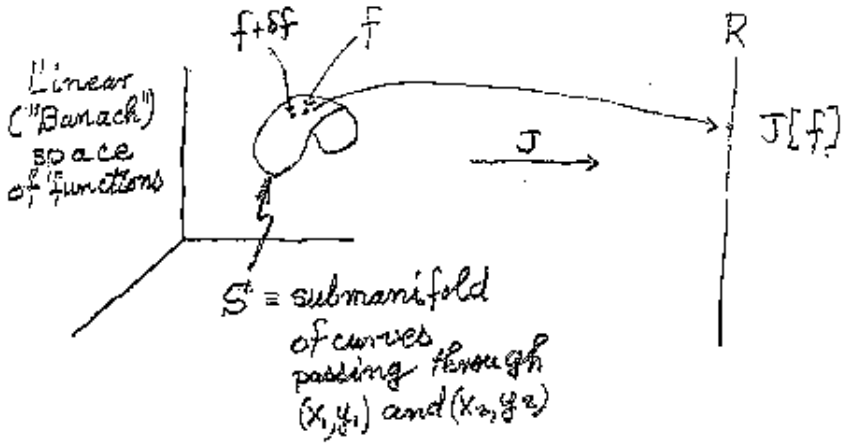


Figure 1.9: The functional J maps its domain space, the submanifold S, into the target space R, the reals

being real-valued, one arrives at the conclusion that, as depicted in Figure 1.9, J is a real-valued functional that maps the submanifold into the reals

$$J: \quad S \rightarrow R = \text{reals} \tag{1.6}$$

$$y \rightsquigarrow J[y] = \int_{x_1}^{x_2} F\left(x; y, \frac{dy}{dx}\right) dx \tag{1.6'}$$

3. Among the set S of admissible functions find the function f which extremizes $J[y]$.

II. The Solution

Based on observations and experiments (ultimately via one's five senses) we have already illustrated the existence of optimal solutions to some typical problems in the calculus of variations. Denoting an optimal solution by f, one mathematize the method for finding it by calculating

$$J[f + \delta f] - J[f] \equiv \Delta J$$

and setting its principal linear part to zero. The calculation is very compact and proceeds as follows:

$$\begin{aligned}\Delta J &= \int_{x_1}^{x_2} \left\{ F(x;y,y')|_{y=f(x)+\delta f(x)} - F(x;y,y')|_{y=f(x)} \right\} dx \\ &= \int_{x_1}^{x_2} \Bigg\{ \cancel{F(x;f,f')} + \frac{\partial F}{\partial y}(x;f,f')\delta f(x) + \frac{\partial F}{\partial y'}(x;f,f') \overbrace{\frac{d(\delta f(x))}{dx}}^{\delta(f'(x))} \\ &\qquad + \begin{pmatrix} \text{higher} \\ \text{order} \\ \text{non-linear} \\ \text{terms} \end{pmatrix} - \cancel{F(x;f,f')} \Bigg\} dx\end{aligned}$$

The Taylor series expansion theorem was used to obtain this result. It depends on the variation δf and its derivative. However, we would like to have the result depend only on δf, and not on its derivative.This desire can be realized by integrating the second term by parts. This means, we recall, that one use $\frac{\partial F}{\partial y'}\frac{d(\delta f)}{dx} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\delta f\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\delta f$. One obtains

$$\Delta J = \underbrace{\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right) \delta f(x) dx + \frac{\partial F}{\partial y'}\delta f(x)\Big|_{x_1}^{x_2}}_{\text{“Principal Linear Part” of } \Delta J} + \begin{pmatrix} \text{higher} \\ \text{order} \\ \text{non-linear} \\ \text{terms} \end{pmatrix} \tag{1.7}$$

The fact that $f + \delta f \in S$ implies that

$$\delta f(x_1) = \delta f(x_2) = 0,$$

which we specified already on page 15. Consequently, the boundary contribution to ΔJ vanishes. We now consider the *Principal Linear Part* of ΔJ, which is linear in δf. We designate it by δJ, and find

$$\delta J = \int_{x_1}^{x_2} \left[F_y - \frac{d(F_{y'})}{dx} \right] \delta f(x) dx. \tag{1.8}$$

This is the *1st variation* of the functional J at $y = f(x,0)$. It is a linear functional which is also called the *differential* of J.

1.4.3 Variational Derivative

The fact that an arbitrary variant of $f(x,0)$ is given by

$$f(x,\alpha_i) = f(x,0) + \sum_{i=1}^{\infty} \alpha_i h_i(x)$$

implies that the variations of f,

$$\delta f(x) = \sum_{i=1}^{\infty} \alpha_i h_i(x) \tag{1.9}$$

form a linear vector space which is spanned by the basis vectors $\{h_i(x) : i = 1, 2, \dots\}$. Taking a cue from multivariable calculus, we can use the differential δJ of J, and identify the *directional derivatives* of J along the respective directions h_k in the space of admissible functions defined on page 15.

These directional derivatives, we recall, are given with the help of

$$\Delta J = \overbrace{J(\alpha, \cdots, \alpha_k, \cdots)}^{J(x_i)} - J(0, \cdots, 0, \cdots),$$

and of Eqs.(1.8) and (1.9) by

$$\begin{aligned} \left.\frac{\partial J(\alpha_i)}{\partial \alpha_k}\right|_{\alpha_i=0} &= \lim_{\alpha_k \to 0} \frac{\Delta J(0, \dots, \alpha_k, \dots)}{\alpha_k} \\ &= \int_{x_1}^{x_2} \left[F_y - \frac{d}{dx}(F_{y'}) \right] h_k(x) dx, \quad k = 1, 2, \dots \end{aligned} \tag{1.10}$$

$$\tag{1.11}$$

In order to establish the parallel with multivariable calculus, let us designate

$$F_y - \frac{d}{dx}\, F_{y'} \equiv \frac{\delta J}{\delta y(x)} \qquad x_1 < x < x_2 \tag{1.12}$$

as the (*first*) *variational derivative of* $J[y]$.

It is clear that this derivative depends on x.

Reminder:

Recall that in multivariable calculus the gradient $\vec{\nabla} f$ of a function is related to its *directional derivative* $D_{\vec{e}_k} f$ along the same basis vector e_k by the equation

$$D_{e_k} f = (\vec{\nabla} f) \cdot \vec{e}_k$$

One may look at the right hand side of the directional derivative, Eq. (1.10), in the same way. It follows that the variational derivative of the functional J,

$$\frac{\delta J}{\delta y(x)} = F_y - \frac{d}{dx}\, F_{y'} \qquad x_1 < x < x_2,$$

corresponds to what in multivariable calculus is the gradient of a function. The obvious difference between the two is that the number of components of a gradient in multivariable calculus is finite and they are labeled by integers. By contrast the "gradient" of a function J, i.e. its variational derivative, has a continuous infinity of components which are labeled by the values of x ($x_1 < x < x_2$).

1.4.4 Euler's Differential Equation

The usefulness of the first variational arises when one considers an "optimal" curve, say

$$y = f(x) \tag{1.13}$$

for which J has a maximal (or minimum, or extremal) value. In such a circumstance all the directional derivates, Eq. (1.10), vanish, i.e.

$$0 = \frac{\partial J(\alpha_i)}{\partial \alpha_k} = \int_{x_1}^{x_2} \frac{\delta J}{\delta y(x)} h_k(x) dx \qquad k = 1, 2, \ldots$$

It follows that for $x_1 < x < x_2$

$$\frac{\delta J}{\delta y(x)} \equiv F_y(x; y(x), y'(x)) - \frac{d}{dx} F_{y'}(x; y(x), y'(x)) = 0. \tag{1.14}$$

This is *Euler's differential equation.* This is the equation which the curve, Eq. (1.13), must satisfy if it extremizes the functional J.

Why must one have $\dfrac{\delta J}{\delta y(x)} = 0$ for $x_1 < x < x_2$? Suppose $\dfrac{\delta J}{\delta y(x)} \neq 0$ in the neighborhood of some point $x = x'$. Then from the set

$$\{h_k(x) : k = 1, 2, \ldots\}$$

we choose or construct a localized "blip" function[5] (see Figure 1.10)

$$\delta f(x) = \sum_{k=1}^{\infty} \beta_k h_k(x) \ .$$

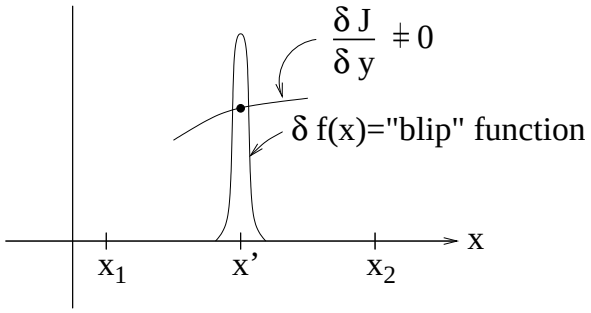


Figure 1.10: A "blip" function is a variation which is differentiable but is nonzero only in a neighborhood around a point x'.

A "blip" function is a type of variation $\delta f(x)$, which is the difference between a trial function $f(x)$ and its variant, say, $f(x) + \delta f(x)$, as in Figure 1.11. Its use would yield

$$0 \neq \int_{x_1}^{x_2} \frac{\delta J}{\delta y(x)} \ \delta f(x) dx = \sum_{k=1}^{\infty} \frac{\partial J}{\partial \alpha_k} \beta_k$$

This contradicts the fact that

$$0 = \frac{\partial J}{\partial \alpha_k} = 0 \qquad \forall \ k.$$

The conclusion is therefore that Euler's differential Eq.(1.14) is indeed the equation which an optimal curve must satisfy. Explicitly, with $F = F(x; y(x), y'(x))$, this equation is

$$\boxed{\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0} \ ,$$

[5] An example of a smooth (i.e. infinitely differentiable) blip function which vanishes for

$$r \equiv |x - x'| \geq \epsilon$$

is

$$\delta f(x) = \begin{cases} \exp\left(-\dfrac{\epsilon^2}{\epsilon^2 - r^2}\right) & \text{for } r < \epsilon \\ 0 & \text{for } r \geq \epsilon. \end{cases}$$

.

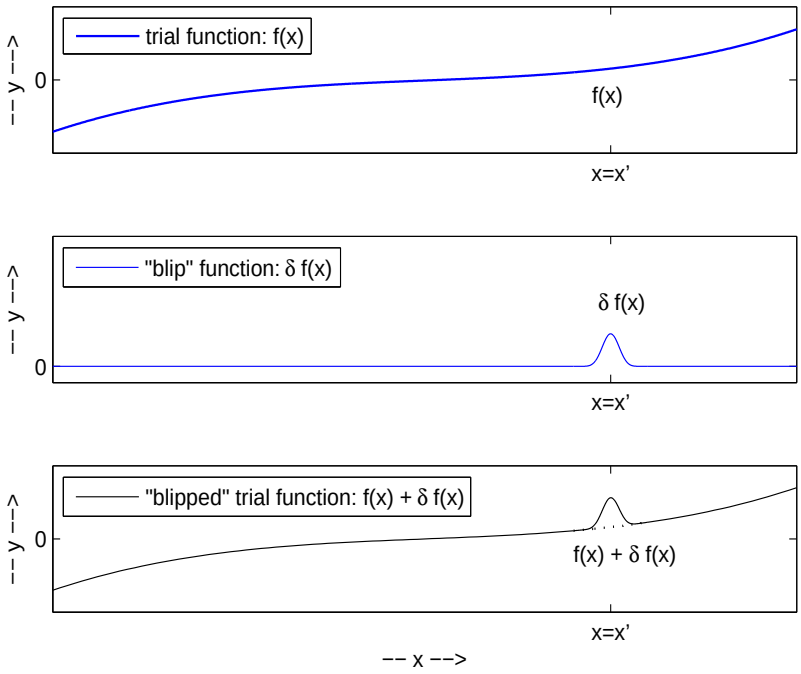


Figure 1.11: The area under any "blip" function equals the area between the trial function $f(x)$ and its variant $f(x) + \delta f(x)$ as depicted in the bottom panel. This area is what in Euler's discrete formulation is the area between a polygonal curve and its variant (the dashed one) in Figure 1.5.

or

$$y'' F_{y'y'} + y' F_{y'y} + F_{y'x} - F_y = 0 \ .$$

This Euler's differential equation is one of second order. It constitutes a necessary condition which $y(x)$ must satisfy if it is to be an optimal curve for the functional $J[y]$.

The Euler equations for the three extremum problems in Section 2 are as follows:

1. Brachistochrone:

$$F = \sqrt{\frac{1 + y'^2}{x}} \qquad \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\ \frac{1}{\sqrt{x}}\right) = 0$$

2. Least area:

$$F = x\sqrt{1 + y'^2} \qquad xy'' + y'(1 + y'^2) = 0$$

3. Fermat:

$$F = n(x, y)\sqrt{1 + y'^2} \qquad ny'' = (n_y - n_x y')(1 + y'^2)$$

1.5 Solved Example

Let us solve the Brachistochrone problem for which $F = \sqrt{\frac{1+y'^2}{x}}$. The fact that $\frac{\partial F}{\partial y} = 0$ implies that the Euler equation is

$$0 = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \ .$$

Consequently

$$\frac{y'}{\sqrt{1+y'^2}} \ \frac{1}{\sqrt{x}} = \text{const} \equiv \left(\frac{1}{2a}\right)^{1/2} \ .$$

Square the result to obtain

$$\frac{1}{1+y'^2} \ \frac{y'^2}{x} = \frac{1}{2a}$$

or

$$y'^2 2a = x + xy'^2 \ .$$

Thus

$$\left(\frac{dy}{dx}\right)^2 = \frac{x}{2a-x} = \frac{x^2}{2ax-x^2} \ . \tag{1.15}$$

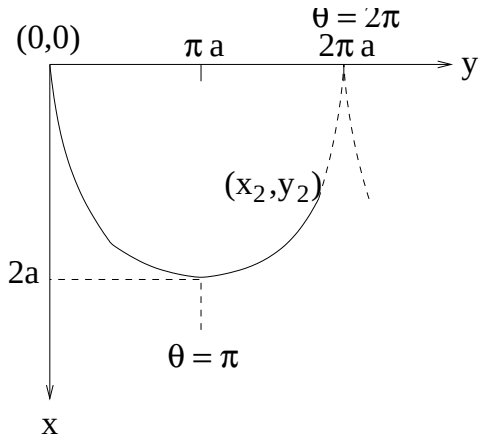


Figure 1.12: The solution curve for the brachistochrone problem is a cycloid.

Equation (1.15) can be integrated. One obtains

$$\begin{aligned} y(x) &= \int_{x_1=0}^{x} \frac{x dx}{(a^2 - a^2 + 2ax - x^2)^{1/2}} \\ &= \int_{x_1=0}^{x} \frac{x dx}{\sqrt{a^2 - (a-x)^2}} \end{aligned}$$

We make the substitution $a - x = a\cos\theta$, or

$$x = a(1 - \cos\theta)$$

to obtain

$$y = \int_{\theta_1}^{\theta} \frac{a(1-\cos\theta)a\sin\theta}{a\sin\theta} d\theta \ .$$

Thus we have

$$\begin{aligned} y &= a(\theta - \sin\theta) + const. \ \leftarrow \text{zero to make } y_1 = 0 \text{ the starting height.} \\ x &= a(1 - \cos\theta) \ . \end{aligned}$$

This is a θ-parametrized cycloid whose amplitude a must be adjusted so that the curve goes through (x_2, y_2).

Lecture 4

1.6 Integration of Euler's Differential Equation.

I. The Brachistochrome problem where

$$F = \sqrt{\frac{1 + y'^2}{x}} \tag{1.16}$$

illustrates the simplification which occurs if the variational integrand F is *independent* of y. In that case

$$\frac{\partial F}{\partial y'} = \text{const.} \tag{1.17}$$

along the curve, and one has to deal only with a 1st order differential equation as was done on page 24. Equation (1.17) is called an *integral of motion.*

II. If F is *independent* of x, as, for example, in Fermat's problem for a stratified atmosphere where

$$F = n(y)\sqrt{1 + y'^2},$$

then an analogous simplification is possible. This is seen best from the "Second Form" of Euler's Equation. Euler's motivational method towards this equation was his desire to obtain an alternate differential equation which contains *no second derivatives of* y whenever his original equation

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 \tag{1.18}$$

was satisfied. To this end he noticed that

$$\frac{dF(x,y,y')}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\ y' + \frac{\partial F}{\partial y'}\ y''$$

and

$$\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = y''\frac{\partial F}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\ .$$

To obtain an expression with no derivatives higher than the first, he subtracted to eliminate the y'' term and obtained

$$\frac{d}{dx}\left(F - y'\ \frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x} + \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\ F_{y'}\right) y'$$

By Euler's equation, Eq.(1.18), the last term vanishes and we are left with the "second form" of Euler's equation

$$\boxed{\frac{d}{dx}\left(F - y'\ F_{y'}\right) = \frac{\partial F}{\partial x}} \tag{1.19}$$

Remark:

1. If the variational integrand F is independent of x, i.e. if $\frac{\partial F}{\partial x} = 0$, then one obtains an *integral of motion*,

   $$F - y'F_{y'} = \text{constant}. \tag{1.20}$$

   This expression involves only the first derivative of the to-be-determined function $y(x)$. It is evident that we have *reduced* the problem from solving a second order differential equation to solving one of first order.

2. By switching independent variables from x to y:

   $$F\left(x, y,\ \frac{dy}{dx}\right)\frac{dx}{dy}\ dy = G\left(y, x, \frac{dx}{dy}\right)\ dy$$

   the Euler equation becomes

   $$0 = \frac{d}{dy}\left(\frac{\partial G}{\partial\left(\frac{dx}{dy}\right)}\right) - \frac{\partial G}{\partial x}\ ,$$

   which implies that

   $$\frac{\partial G}{\partial\left(\frac{\partial x}{\partial y}\right)} = const.$$

   is an integral of motion. Thus there is an alternate way of arriving at the integral of motion (1.20).

Example 1 Fermat's Principle for a Stratified Atmosphere.

The variational integrand for least time propagation in an x-translation invariant but y-stratified medium is

$$F = n(y)\sqrt{1+y'^2}.$$

The corresponding E-L equation is a differential equation of second order, and because it *is* of second order, the method of solving it is a two-stage process.

Stage 1.

In light of the second form of Euler's equation, the first stage reduces to a single step: by inspection obtain a first integral of motion,namely

$$F - y'F_{y'} = n(y)\left[\sqrt{1+y'^2} - \frac{y'^2}{\sqrt{1+y'^2}}\right] = \frac{n(y)}{\sqrt{1+y'^2}} = constant \equiv c$$

The result is a c-parametrized family of first order differential equations

$$\frac{dy}{dx} = \sqrt{\frac{n^2}{c^2} - 1}\ .$$

From its right hand side one draws a key conclusion,

$$\boxed{\text{No real solution exists whenever } n^2(y) < c^2.}$$

This mathematizes the fact that, for an integral of motion satisfying this inequality, no propagation is posible.

To concretize this feature, and to exhibit a type of stratified Fermat problem which is explicitly soluble in mathematically closed form, consider propagation in an exponential atmosphere. There the refractive index has the form

$$n = n_0 e^{-y}.$$

Focus on propagation which starts at the origin ($x = 0, y = 0$). There the initial data is

$$\left.\frac{dy}{dx}\right|_{x=0} = y'_0 \equiv \tan\alpha \text{ (“initial slope”)}, \tag{1.21}$$

$$n(y=0) = n_0 \qquad \text{(“refractive index at ground level”)}. \tag{1.22}$$

This initial data determines the integral of motion:

$$c = \frac{n_0}{\sqrt{1+y'^2_0}} = n_o \cos\alpha, \tag{1.23}$$

and the equation governing the corresponding unique motion is

$$\left(\frac{dy}{dx}\right) = \pm\sqrt{\frac{n^2(y)}{c^2} - 1} = \pm\sqrt{\frac{e^{-2y}}{\cos^2\alpha} - 1}. \tag{1.24}$$

The upper sign mathematizes the upward progress of the trajectory $y(x)$, the lower sign its downward progress. These two branches of $y(x)$ meet at the turning point where their slopes vanish.
Thus the first stage of the solution process yields, among others, the result the maximum height y_{max} is achieved when $\frac{dy}{dx} = 0$:

$$e^{-y_{max}} = \cos\alpha \tag{1.25}$$

or

$$y_{max} = -\log\cos\alpha. \tag{1.26}$$

This maximum height depends on the launching angle α:

$$\alpha = 0 \Rightarrow \ y_{max} = 0 \tag{1.27}$$

$$\alpha \to \frac{\pi}{2} \Rightarrow \ y_{max} \to -\infty \tag{1.28}$$

Stage 2.

That being is the case, what is the trajectory's return distance d? How does it depend on the initial angle α in Figure 1.13? The answer to these questions lies in the second integral obtained by integrating the differential Eq.(1.24). Upon solving this differential equation by separating variables, using the properties of elementary functions, and introducing the appropriate integration limits, one finds that

$$d = 2\alpha, \quad \text{where } \cos\alpha = e^{-y_{max}}. \tag{1.29}$$

Example 2. Geodesic on a Sphere.

A geodesic is a line which represents the shortest path between two given points when the path is restricted to lie on some surface. On a sphere of radius ρ the square element of length is

$$(ds)^2 = \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\varphi^2$$

Hence the length of a path $\theta(\varphi)$ between two points, say 1 and 2, is

$$s = \int_1^2 ds = \rho \int_1^2 \sqrt{\left(\frac{d\theta}{d\varphi}\right)^2 + \sin^2\theta(\varphi)}\ d\varphi$$

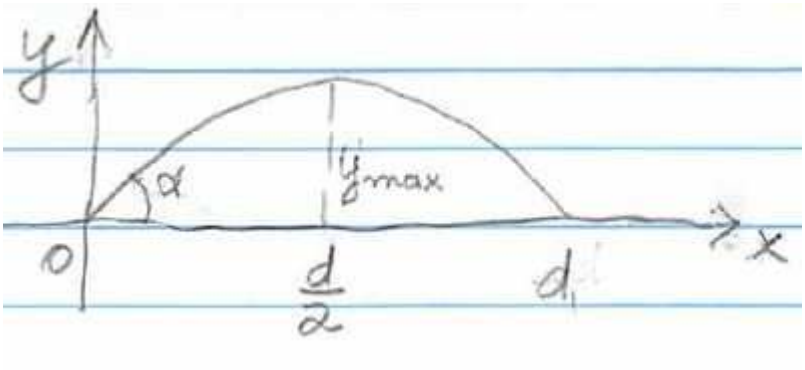


Figure 1.13: Trajectory of light ray in an exponentially stratified medium

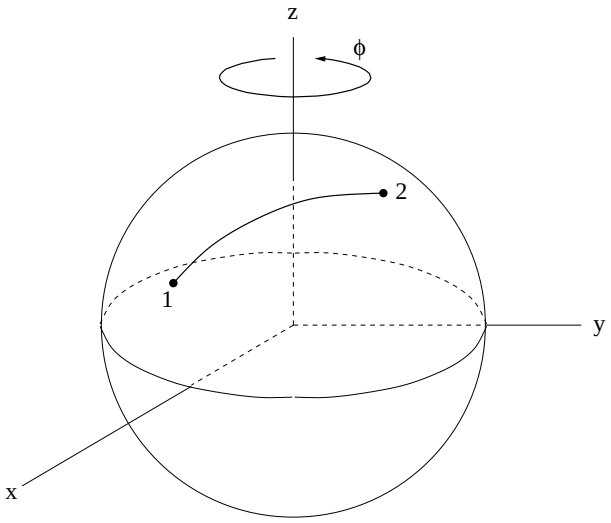


Figure 1.14: Curve on a two-sphere of radius ρ.

Here the variational integrand is

$$F = \sqrt{\theta_\varphi^2 + \sin^2\theta}$$

Note that F is independent of φ. Consequently, one can apply the "second form" of Euler equation. One finds that

$$F - \theta_\varphi \frac{\partial F}{\partial \theta_\varphi} = \text{constant} \equiv a.$$

Explicitly one has

$$\sqrt{\theta_\varphi^2 + \sin^2\theta} - \theta_\varphi \frac{\theta_\varphi}{\sqrt{\theta_\varphi^2 + \sin^2\theta}} = a$$

This is a differential equation which one readily solves in the usual way: Multiply by $\sqrt{\quad}$ and cancel θ_φ^2 to obtain

$$\sin^2\theta = a\sqrt{\left(\frac{d\theta}{d\varphi}\right)^2 + \sin^2\theta} \tag{1.30}$$

Isolate the derivative by solving for it. One obtains

$$\begin{aligned}\frac{d\varphi}{d\theta} &= \left(\frac{\sin^4\theta}{a^2} - \sin^2\theta\right)^{-1/2} \\ &= \frac{\csc^2\theta}{\sqrt{\frac{1}{a^2} - \csc^2\theta}} = \frac{\csc^2\theta}{\sqrt{\beta^2 - \cot^2\theta}}.\end{aligned}$$

Here $\beta^2 = \frac{1}{a^2} - 1$. The antiderivative is

$$\varphi(\theta) = -\sin^{-1}\frac{\cot\theta}{\beta} + \alpha\ .$$

Consequently, one obtains

$$-\cot\theta = \beta\sin(\varphi - \alpha)$$

This is the equation of a plane through the origin: Multiply by $\rho\sin\theta$ and expand

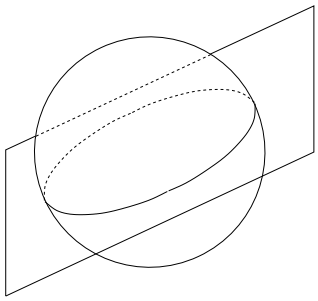


Figure 1.15: A great circle is the intersection of a sphere with a plane through the origin.

the sine function,

$$-\underbrace{\rho\cos\theta}_{z} = \beta\underbrace{\rho\sin\theta\sin\varphi}_{y}\cos\alpha - \beta\underbrace{\rho\sin\theta\cos\varphi}_{x}\sin\alpha$$

$$\boxed{z + y\,\beta\cos\alpha - x\,\beta\sin\alpha = 0}$$

Thus, the geodesic is the locus of the sphere intersected by a plane. In other words, the geodesic is a segment of a great circle.

Remark: The variational integral is extremized by *many* solutions to the Euler equation. The shortest geodesic minimizes s. Other solutions are geodesics which consist of great circles that wind around the sphere. They constitute *saddle points* of s in the space of functions!

Chapter 2

GENERALIZATIONS

2.1 Functional with Several Unknown Functions

Consider a variational integral which is a functional of several functions with the value of the functions fixed at the endpoints. The extremization of such an integral, namely

$$J[y_i] = \int_{x_1}^{x_2} F(x; y_1, y_1', y_2, y_2', \ldots, y_n, y_n')dx,$$

where $y_i(x)$ $i = 1, \ldots, n$ satisfy the prescribed boundary conditions at $x = x_1$ and $x = x_2$, is achieved by a method analogous to the one described before. Thus, we consider a system of variants

$$y_j = y_j(x, 0) + \sum_{i=1} \alpha_i^j h_i(x) = f_j(x, 0) + \sum_{i=1} \alpha_i^j h_i(x) j = 1, \ldots, n$$

of the system of functions

$$y_j = y_j(x, 0) = f_j(x, 0) \quad j = 1, \ldots, n$$

which we assume extremizes the functional $J[y_i]$.

The fact that these functions and their variants have prescribed values at the endpoints x_1 and x_2 can be stated by saying that each of the α-parametrized curves,

$$\{y_j = f_j(x, \alpha_i^j), j = 1, \ldots n\} \ ,$$

passes through the same two end points. This means that the system of variations

$$\delta f_j(x) \quad j = 1, \ldots, n \ ,$$

which can be pictured as an n-component vector, vanishes at the endpoints:

$$\delta f_j(x_1) = \delta f_j(x_2) = 0 \quad j = 1, \ldots, n \ .$$

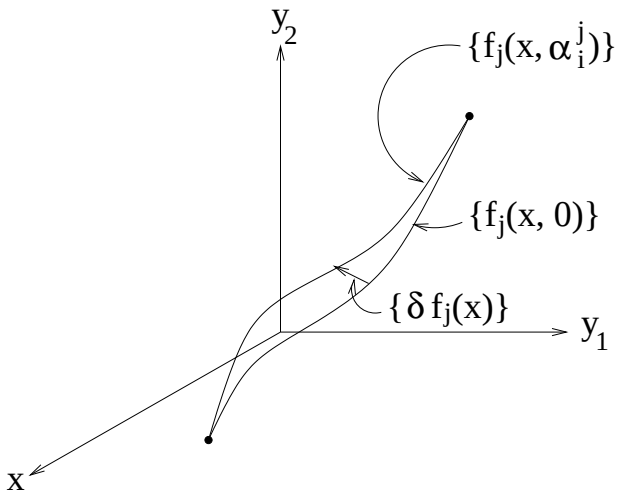


Figure 2.1: A variant and its system of variations $\delta\{f_j(x)\}$

The procedure for obtaining the necessary condition for J to be an extremum follows the same pattern as before. The first variation of the functional J is

$$\delta J = \sum_{j=1}^{n} \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y_j}\ \delta f_j + \frac{\partial F}{\partial y_j'}\ \delta\left(\frac{df_j}{dx}\right) \right\} dx$$

Using the fact that

$$\begin{aligned} \delta\left(\frac{df_j}{dx}\right) &= \frac{df_j(x,\alpha_i^j)}{dx} - \frac{df_j(x,0)}{dx} \\ &= \frac{d}{dx}\ \delta f_j(x) \quad j = 1,\ldots,n \end{aligned}$$

and integrating by parts yields

$$\delta J = \sum_{j=1}^{n} \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial y_j} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_j'}\right) \right\} \delta f_j(x) dx + \frac{\partial F}{\partial y_j'}\ \delta f_j \Big|_{x_1}^{x_2}.$$

The boundary term vanishes. The fact that for J to be an extremum for arbitrary variations $\delta f_j\ j = 1,\ldots,n$, implies that

$$\frac{\delta J}{\delta y_j(x)} \equiv F_{y_j} - \frac{d}{dx}(F_{y_j'}) = 0 \quad j = 1,\ldots,n \tag{2.1}$$

the Euler equations for "a system with n degrees of freedom". This set of equations constitutes the necessary condition for the variational integral $J[y_1,\ldots,y_n]$ to be an

extremum. It is evident that this condition generalizes Euler equation from a single degree of freedom, Eq.(1.14) on page 21, to many degrees of freedom.

There is, however, one qualitatively new feature. It may happen that the vanishing of the first variation of J is due to the fact that J is an *maximum* with respect to variations in, say, $y_1(x)$ but is *minimum* with respect to variations in, say, $y_2(x)$.

Whatever the case may be, we shall call a curve $\{y_1(x), \ldots, y_n(x)\}$ which solves the set of Euler equations (2.1) an "*extremal curve*", even though often an "*optimal curve*" may be more descriptive.

Example (Fermat's Principle in three dimensions).

The propagation of a beam of light or sound whose local propagation speed is

$$v = \frac{c}{n(x, y, z)}$$

is governed by the extremum principle

$$J[y, z] = c \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2 + z'^2}}{n(x, y, z)} dx = \text{Minimum}.$$

This principle assumes that the local propagation speed depends only on the position (x, y, z) of the propagating beam. In that case the propagation medium in said to be *inhomogeneous* but *isotropic*. If, however, the propagation also depends on the direction of the beam, i.e.

$$v = \frac{c}{n(x, y, z, y', z')}$$

then the medium is said to be *inhomogeneous* and *anisotropic*.

In that case the propagation law is given by

$$J[y, z] = c \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2 + z'^2}}{n(x, y, z, y', z')} \, dx = \text{Minimum}.$$

Example (Geodesic on a Sphere)

Consider the collection of all paths $(\theta(\lambda),\ \varphi(\lambda))$ joining two fixed points on the two-sphere. The parameter λ may be any variable which expresses monotonic progress along the path. For example, λ = time t of a moving ship on the earth, azimuthal angle φ, arclength s, or any other curve parameter monotonically related to the others.

The task is to find the λ-parametrized path $(\theta(\lambda),\ \varphi(\lambda))$ on the two sphere which has the property that its length

$$J = \int \sqrt{\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta \left(\frac{d\varphi}{d\lambda}\right)^2} \, d\lambda = \text{Extremum} \ .$$

For this problem one must find two functions that satisfy two differential equations:
(i) $$\frac{d}{d\lambda}\left(\frac{\partial F}{\partial \theta'}\right) = \frac{\partial F}{\partial \theta}$$
or
$$\frac{d}{d\lambda}\ \frac{\frac{d\theta}{d\lambda}}{\sqrt{\theta'^2 + \sin^2\theta\varphi'^2}} = \frac{\sin\theta\cos\theta}{\sqrt{\theta'^2 + \sin^2\varphi'^2}}\ \left(\frac{d\varphi}{d\lambda}\right)^2 \tag{2.2}$$
(ii) The second equation is
$$\frac{d}{d\lambda}\left(\frac{\partial F}{\partial \varphi'}\right) = \frac{\partial F}{\partial \varphi}$$

The fact that the *variational integrand* F is independent of φ (one says that F is "cyclic" in φ) means that
$$\frac{\partial F}{\partial \varphi} = 0$$
Consequently, we have immediately an integral of motion, namely
$$\frac{\partial F}{\partial \varphi'} = \frac{\varphi' \sin^2\theta}{\sqrt{\theta'^2 + \sin^2\theta\varphi'^2}} = const. \equiv P_\varphi\ . \tag{2.3}$$

Equation (2.2) and Eq. (2.3) constitute the system of ordinary differential equations which we must solve. If we let $\lambda = \varphi$ be the curve parameter, then Eq.(2.3) becomes Eq.(1.30) (on page 30) and we can solve the problem. However, if the variational integrand had been
$$F \ = \ \left[g_{11}\left(\frac{dy_1}{d\lambda}\right)^2 + g_{22}\left(\frac{dy_2}{d\lambda}\right)^2 + 2g_{12}\frac{dy_1}{d\lambda}\ \frac{dy_2}{d\lambda}\right]^{1/2},$$
where the g_{ij}'s are functions which depend on the coordinates y_1, and y_2, then we could not have solved the problem. Nevertheless, a crisp and simple conclusion is possible.

What one does is to introduce the arclength as the curve parameter:
$$s = \int_0^\lambda \sqrt{\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2}\ d\lambda$$
or
$$ds = \sqrt{\theta'^2 + \sin^2\theta\varphi'^2}\ d\lambda\ .$$
This brings about a tremendous simplification. The arclength parameter is introduced as follows
$$\frac{d}{ds} = \frac{d\lambda}{ds}\ \frac{d}{d\lambda} = \frac{1}{\sqrt{\theta'^2 + \sin^2\theta\varphi'^2}}\ \frac{d}{d\lambda}\ . \tag{2.4}$$

One applies this differential operator to θ and obtains

$$\frac{1}{\sqrt{\quad}}\,\frac{d\theta}{d\lambda} = \frac{d\theta}{ds}\ .$$

Indicating differentiation with respect to s by a dot, $\frac{d\theta}{ds} = \dot{\theta}$, one obtains for Eq.(2.2)

$$\frac{1}{\sqrt{\quad}}\,\frac{d\dot{\theta}}{d\lambda} = \sin\theta\cos\theta\left(\frac{1}{\sqrt{\quad}}\,\frac{d\varphi}{d\lambda}\right)^2\ .$$

Applying the above differential operator to $\dot{\theta}$ and to φ twice, one obtains

$$\ddot{\theta} = \dot{\varphi}^2\sin\theta\cos\theta\ .$$

Similarly, the constant of motion, Eq.(2.3) becomes

$$P_\varphi = \dot{\varphi}\sin^2\theta\ .$$

If s is proportional to time, then in mechanics this is the "angular momentum" around the z-axis.

The introduction of the arclength s as the curve parameter always implies that

$$\left(\frac{d\theta}{ds}\right)^2 + \sin^2\left(\frac{d\varphi}{ds}\right)^2 = constant$$

along a curve. This expression is an integral of motion even for curves which are not extremals. The availability of this integral is guaranteed without having to integrate the Euler equations for the variational problem. Indeed, one always has

$$1 = \frac{ds}{ds} = \frac{\sqrt{d\theta^2 + \sin^2 d\varphi^2}}{ds} = \sqrt{\left(\frac{d\theta}{ds}\right)^2 + \sin^2\left(\frac{d\varphi}{ds}\right)^2}\ . \tag{2.5}$$

The constancy of this quantity will be discussed and used again on page 51, but this time in the context of a *constrained* variational problem.
Remarks:

1. The arclength parameter *always* simplifies the parametric representation in this way.

2. Equation (2.5) constitutes an integral of motion:

$$1 = \left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{ds}\right)^2\ .$$

It is a constant of motion along *any curve*, geodesic or non-geodesic on S^2, provided s is the arclength parameter. In other words,

$$\frac{d}{ds}\left[\sqrt{\left(\frac{d\theta}{ds}\right)^2+\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2}\right]=0$$

along *any* curve. Thus there always is an "energy type" integral

$$\frac{1}{2}\left[\left(\frac{d\theta}{ds}\right)^2+\sin^2\theta\left(\frac{d\varphi}{ds}\right)^2\right]=const.$$

The same conclusion holds in the general context as expressed by the variational integrand given by Eq. (2.4).

The conclusion is this: For a geodesic we have

$$\begin{aligned}
&\ddot{\theta}=\dot{\varphi}^2\sin\theta\cos\theta \\
&\dot{\varphi}\sin^2\theta=const. && \text{"angular momentum"} \\
&\frac{1}{2}(\dot{\theta}^2+\sin^2\theta\dot{\varphi}^2)=\frac{1}{2} && \text{"energy"}
\end{aligned}$$

3. These three equations in two unknowns are not independent. Instead, any *two* equations imply the *third*. For example, $\frac{d}{ds}(\dot{\theta}^2+\sin^2\theta\dot{\varphi}^2)=0$ because of the two Euler-Lagrange equations. The benefit is that one can grasp the trajectories on a two-sphere in terms of the two simple integrable equations,

$$\frac{1}{2}\left(\dot{\theta}^2+\frac{(const.)^2}{\sin^2\theta}\right)=constant$$

and

$$\dot{\varphi}\sin^2\theta=const.\ ,$$

whose solutions are arclength-parametrized curves.

Lecture 5

2.2 Extremum Problem with Side Conditions.

We would like to consider a class of problems in the calculus of variations, the most famous of which is the iso-perimetric problem; namely, find a curve such that

$$\text{Area}=\iint dxdy=\oint ydx=J[y]=\text{extremum}$$

subject to the constraint that the bounding curve $y(x)$ has fixed length

$$\ell = \oint \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \text{constant}.$$

An interesting generalization of this problem was the one which confronted Dido.

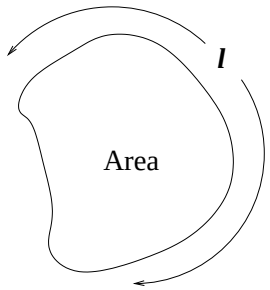


Figure 2.2: Area having perimeter of length ℓ.

Her problem was to circumscribe by means of a curve of *fixed length* an area which yields the largest amount of fertility. In other words, for a given function $\rho(x, y)$ (say, yield per unit area), extremize the weighted area integral

$$\iint \rho(x, y) dx dy$$

subject to the condition that the bounding curve $y(x)$ has fixed length, $\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \equiv \ell = \text{constant}$.

More generally, the problem is to find in the space of functions that function $y(x)$ for which $J[y] \equiv \int_{x_1}^{x_2} F(x; y, y')dx = \text{extremum}$, but which at the same time satisfies the constraints

$$\begin{aligned} K_1[y] &\equiv \int_{x_1}^{x_2} G_1(x; y, y')dx = \text{constant} \equiv c_1 \\ &\vdots \\ K_k[y] &\equiv \int_{x_1}^{x_2} G_k(x; y, y')dx = \text{constant} \equiv c_k \end{aligned} \tag{2.6}$$

2.2.1 Heuristic Solution

We can solve this optimization ("extremum") problem with constraints by setting up the necessary conditions the desired function $y(x)$ must satisfy. As usual, we consider the variants,

$$y(x) + \delta y(x)$$

of $y(x)$. Each variant is uniquely characterized by an infinitesimal vector, the variation $\delta y(x)$ of $y(x)$. The resulting variations in J and K_i are

$$\begin{aligned} \delta J &= J[y + \delta y] - J[y] \equiv \int_{x_1}^{x_2} \frac{\delta J}{\delta y(x)}\, \delta y(x) dx \\ \delta K_i &= K_i[y + \delta y] - K[y] \equiv \int_{x_1}^{x_2} \frac{\delta K_i}{\delta y(x)}\, \delta y(x) dx \end{aligned}$$

There are two types of variants $y(x) + \delta y(x)$:

I. those which do and

II. those which do not

satisfy the given constraints.

I. Those which do satisfy the contraints have variations which are perpendicular to the gradient of the constraint surfaces $K_i = c_i (i = 1, \ldots, k)$, i.e. those for which

$$\begin{aligned} \int_{x_1}^{x_2} \frac{\delta K_i}{\delta y(x)}\, \delta y(x) dx &= 0 \quad \forall i = 1, \ldots, k \\ K_i[y] &= c_i \quad i = 1, \ldots, k_1 \end{aligned} \tag{2.7}$$

We shall call such vectors the "yes" vectors. They are tangent to all the constraint manifolds.

II. Those variants which do *not* satisfy the constraints have variations which are *not* perpendicular to the constraint surfaces $K_i = c_i$, i.e. those for which

$$\int_{x_1}^{x_2} \frac{\delta K_i}{\delta y(x)}\, \delta y(x) dx \neq 0 \quad \text{for some } i = 1, \ldots, k$$

We shall call such vectors the "no" vectors.

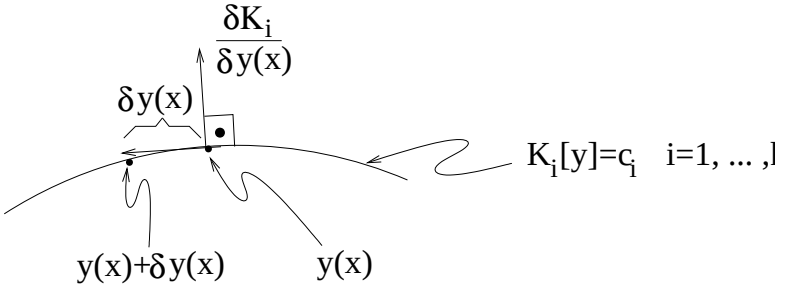


Figure 2.3: ith constraint manifold: its gradient $\frac{\delta K_i}{\delta y(x)}$ which is perpendicular to any class I ("yes") vector $\delta y(x)$ which is tangent to that manifold, $K_i[y] = c_i$.

Thus[1]a variation vector $\delta y(x)$ always belongs either to class I ("yes") or to class II ("no"): If δy belongs to class I, then

$$\delta J = \int_{x_1}^{x_2} \frac{\delta J}{\delta y(x)}\, \delta y(x) dx = 0$$

because $y(x)$ was assumed to be an optimal function which satisfies the constraint, i.e. for which

$$\delta K_i = \int_{x_1}^{x_2} \frac{\delta K_i}{\delta y(x)}\, \delta y(x) dx = 0 \quad i = 1, \ldots, k \text{ (for class I variants).}$$

If, on the other hand, δy belongs to class II, i.e. such that

$$\delta K_i \equiv \int \frac{\delta K_i}{\delta y(x)}\, \delta y(x) dx \neq 0 \quad \text{for some } i = 1, \ldots, k,$$

then we can, and we will, choose constants $\lambda_1, \ldots, \lambda_k$ such that

$$\delta J = \lambda_1 \delta K_1 + \ldots \lambda_k \delta K_k.$$

The conclusion is therefore this: *all* variation vectors $\delta y(x)$ satisfy the equation

$$\int_{x_1}^{x_2} \left[\frac{\delta J}{\delta y(x)} - \lambda_1 \frac{\delta K_1}{\delta y(x)} - \ldots \lambda_k \frac{\delta K_k}{\delta y(x)} \right] \delta y(x) dx = 0$$

whenever

$$J - \lambda_1 K_1 - \cdots - \lambda_k K_k = \text{extremum}$$

[1]Following Aristotle's law of the excluded middle.

where the constants $\lambda_1, \ldots, \lambda_k$ *must be chosen* appropriately. They are determined by the values $c_1, \ldots, c_k$ of the constraints.

Summary:

To solve the optimization problem with constraints, solve the following $k+1$ equations

$$\begin{aligned} \frac{\delta}{\delta y(x)} \left(J[y] - \lambda_1 K_1[y] - \ldots \lambda_k K_k[y]\right) &= 0 \\ K_1[y] &= c_1 \\ &\vdots \\ K_k[y] &= c_k \end{aligned}$$

for $y(x)$ and $\lambda_1, \ldots, \lambda_k$. The constants $\lambda_1, \ldots, \lambda_k$ are called "Lagrange multipliers".

2.2.2 Solution via Constraint Manifold

The "yes" and "no" vectors play a basic role in establishing necessary conditions for the existence of an extremum subject to constraints. It is therefore appropriate to highlight how these vectors tie into well-established theory. We shall do this by splitting the space of tangent vectors into two orthogonal vector spaces, whose only common vector is the zero vector.

Finite-dimensional Space

Our task is to extremize a given function $f(\vec{x})$ on the finite-dimensional space R^n subject to a finite number of constraints, say, $g_1(x) = c_1, \cdots, g_k(x) = c_k$, with $k < n$. More precisely, we wish to find the equations for those critical points $\vec{x}_0$ of $f(\vec{x})$ which lie on the constraint manifold, which is the intersection of the constraint hypersurfaces given by the finite number of constraints.

We shall solve this problem in two steps. First, we shall develop the chain of reasoning when there are no constraints present. Then we shall extend this reasoning to the case where k constraints are present.

Extremum Without Constraints

Consider the trajectory $\vec{x} = \vec{r}(t)$ of a particle in the conservative force field ∇f of a potential $f(\vec{x})$. Then

$$\frac{df(\vec{r}(t))}{dt} = \sum_i \frac{\partial f}{\partial x^i}\frac{dr^i}{dt} \equiv \nabla f \cdot \frac{d\vec{r}}{dt} = \text{"(-)force} \times \text{velocity"} \tag{2.8}$$

is the rate at which the force field does work on the particle.

Suppose there exists a point $\vec{x}_0$ such that

$$\nabla f(\vec{x}_0) \cdot \frac{d\vec{r}}{dt} = 0$$

for the tangent vectors of *all* curves $\vec{x} = \vec{r}(t)$ passing through this point, or, more generally, for all vectors $\vec{v}$ such that

$$\nabla f(\vec{x}_0) \cdot \vec{v} = 0.$$

These vectors form the *tangent space* at $\vec{x}_0$. Observe that

$$\nabla f(\vec{x}_0) \cdot \vec{v} = 0$$

implies that

$$\nabla f(\vec{x}_0) \in \text{“orthogonal complement” of the tangent space at } \vec{x}_0 \ .$$

Being the orthogonal complement of the space of *all* tangent vectors, it consists of the zero vector only; it is zero dimensional. Thus

$$\nabla f(\vec{x}_0) = \vec{0} \ .$$

This is the necessary condition for $\vec{x}_0$ to be the location of an unconstrained extremum of f.

Extremum with Constraints

Next, suppose there exists a point $\vec{x}_0$ such that

$$\nabla f(\vec{x}_0) \cdot \frac{d\vec{r}}{dt} = 0 \tag{2.9}$$

for the tangent vectors $\frac{d\vec{r}}{dt}$ of only those curves $\vec{x} = \vec{r}(t)$ passing through $\vec{x}_0$ which lie in the *constraint manifold*

$$\{\vec{x} : \ g_1(\vec{x}) = c_1, \cdots, g_k(\vec{x}) = c_k\} \ , \tag{2.10}$$

and hence satisfy

$$\nabla g_1(\vec{x}_0) \cdot \frac{d\vec{r}}{dt} = 0, \cdots, \nabla g_k(\vec{x}_0) \cdot \frac{d\vec{r}}{dt} = 0 \ . \tag{2.11}$$

The collection of such vectors $\frac{d\vec{r}}{dt}$ forms a *subspace* of the set of all vectors at x_0: any linear combination of the subspace vectors also satisfies these k constraints. This subspace,

$$W = \{\vec{v} : \nabla g_1(\vec{x}_0) \cdot \vec{v} = 0, \cdots, \nabla g_k(\vec{x}_0) \cdot \vec{v} = 0\} \tag{2.12}$$

is called the *constraint subspace* at $\vec{x}_0$. These subspace vectors are the "yes" vectors for the infinite dimensional domain in Eq.(2.7) on page 40.

One now makes the fundamental and final observation that $\nabla f(\vec{x}_0)$ belongs to the the orthogonal complement of the constraint subspace W, Eq.(2.12). This orthogonal complement consists precisely of those vectors which have the form $\lambda_1 \nabla g_1 + \cdots + \lambda_k \nabla g_k$. In particular,

$$\nabla f(\vec{x}_0) = \lambda_1 \nabla g_1 + \cdots + \lambda_k \nabla g_k \tag{2.13}$$

for some set of coefficients $\lambda_1, \cdots, \lambda_k$. They are called the *Lagrange multipliers* of the constrained extremum problem. It is evident that their values and the location of the critical point of f on the constraint manifold are given by the following "Lagrange Multiplier" theorem

Theorem 1.

Given: *Let f be a scalar function which has an extremum at $\vec{x}_0$ on the intersection of the k constraint hypersurfaces ("the constraint manifold"), Eq.(2.10).*

Conclusion: *The location of the constrained critical point is determined by*

$$\boxed{\nabla f(\vec{x}_0) - \lambda_1 \nabla g_1(\vec{x}_0) - \cdots - \lambda_k \nabla g_k(\vec{x}_0) = 0\}} \tag{2.14}$$

where the coefficients $\lambda_1, \cdots, \lambda_k$ are obtained in conjunction with

$$\begin{aligned} g_1(\vec{x}_0) &= c_1 \\ &\vdots \\ g_k(\vec{x}_0) &= c_k \end{aligned}$$

Remark 1: In mechanics where $f(\vec{x})$ is a scalar potential and $\nabla f(\vec{x})$ its (conservative) force field, the force

$$\nabla f(\vec{x}_0) = \lambda_1 \nabla g_1(\vec{x}_0) + \cdots \lambda_k \nabla g_k(\vec{x}_0)$$

is the constraining force which guarantees that the point particle stays in equilibrium ("no net forces") on the constraint manifold.
Remark 2: This theorem is easily visualized in three dimensions, $R^n = R^3$, with two constraints, say $g_1(\vec{x}) = c_1$ and $g_2(\vec{x}) = c_2$. There the constraint manifold is one-dimensional, a single curve $\vec{r}(t)$, the intersection of the two surfaces $g_1(\vec{x}) = c_1$ and $g_2(\vec{x}) = c_2$. Consequently,

$$\begin{aligned} \vec{\nabla} g_1 \cdot \frac{d\vec{r}}{dt} &= 0 \\ \vec{\nabla} g_2 \cdot \frac{d\vec{r}}{dt} &= 0 \end{aligned}$$

The fact that f has an extremum on this curve implies

$$\vec{\nabla} f \cdot \frac{d\vec{r}}{dt} = 0$$

Consequently, $\exists \lambda_1$ and λ_2 such that

$$\nabla f = \lambda_1 \vec{\nabla} g_1 + \lambda_2 \vec{\nabla} g_2.$$

Example (Function with two finite constraints):
In multivariable calculus the problem

$$f(x, y, z) = \text{extremum}$$

subject to

$$\begin{aligned} g(x, y, z) &= c_1 \\ h(x, y, z) &= c_2 \end{aligned}$$

is solved by considering

$$f^* = f - \lambda_1 g - \lambda_2 h$$

and solving

$$\begin{aligned} \vec{\nabla}(f - \lambda_1 g - \lambda_2 h) &= 0 \\ g(x, y, z) &= c_1 \\ h(x, y, z) &= c_2 \end{aligned}$$

for x_0, y_0, z_0, λ_1 and λ_2. These are 3+2 equations in 3+2 unknowns.

Infinite-dimensional Space

The Lagrange Multiplier theorem readily extends to a functional which is defined on curves in function space, which is infinite-dimensional. The development runs parallel to the one preceeding Theorem 1.

First of all recall that a "curve in function space" is a 1-parameter set of functions

$$y(x,t) \qquad a \le t \le b \ . \tag{2.15}$$

Second, the rate of change of the functional $J[y]$ along this 1-parameter set is

$$\begin{aligned}
\frac{d}{dt}J[y] &= \frac{d}{dt}\int_{x_1}^{x_2} F(x; y(x,t), y'(x,t))dx \\
&= \int_{x_1}^{x_2} \left[\left.\frac{\partial F}{\partial y}\right|_{y(x,t)} \frac{\partial y(x,t)}{\partial t} + \left.\frac{\partial F}{\partial y'}\right|_{y(x,t)} \frac{\partial y'(x,t)}{\partial t} \right] dx \\
&= \int_{x_1}^{x_2} \left.\frac{\delta J}{\delta y(x)}\right|_{y(x,t)} \frac{\partial y(x,t)}{\partial t} dx \ ,
\end{aligned}$$

where $\delta J/\delta y(x)$ is the variational derivative of J as defined by Eq.(1.12) on page 20.

Third, if $J[y]$ has an extremal value on the parametrized set of functions $y(x,t)$ at $t = t_0$, which is to say that its rate of change $dJ[y]/dt$ vanishes at $t = t_0$, then

$$0 = \left.\frac{d}{dt}J[y]\right|_{t=t_0} = \int_{x_1}^{x_2} \left.\frac{\delta J}{\delta y(x)}\right|_{y(x,t_0)} \left.\frac{\partial y(x,t)}{\partial t}\right|_{t_0} dx \ . \tag{2.16}$$

replaces Eq.(2.9) as the extremum condition.

Fourth, the fact that the one-parameter family $y(x,t)$ lies in the intersection of the constraints, Eqs.(2.6) on page 39, implies that

$$\begin{aligned}
\frac{dK_1[y]}{dt} &= \int_{x_1}^{x_2} \left.\frac{\delta K_1}{\delta y(x)}\right|_{y(x,t)} \frac{\partial y(x,t)}{\partial t} dx = 0 \\
&\vdots \\
\frac{dK_k[y]}{dt} &= \int_{x_1}^{x_2} \left.\frac{\delta K_k}{\delta y(x)}\right|_{y(x,t)} \frac{\partial y(x,t)}{\partial t} dx = 0 \ .
\end{aligned} \tag{2.17}$$

At $t = t_0$ these equations replace Eqs.(2.11).

Finally, Eq.(2.13) gets replaced by

$$\frac{\delta J}{\delta y(x)} = \lambda_1 \frac{\delta K_1}{\delta y(x)} + \cdots + \lambda_k \frac{\delta K_k}{\delta y(x)} \ .$$

Nota bene: A class I ("yes") or a class II ("no") vector $\delta y(x)$ on page 40 is an infinitesimal vector ("a variation")

$$\delta y(x) = \frac{\partial y(x,t)}{\partial t}\delta t$$

tangent to one of the t-parametrized sets of functions ("curve in function space"), Eq.(2.15), on page 46. A curve which satisfies the integral constraints has tangent vectors which obey Eq.(2.17). These tangents are proportional to the "yes" vectors. The orthogonal complement of all such vectors make up the vector space of "no" vectors. Each of them is some linear combination of $\delta K_1/\delta y(x), \cdots, \delta K_k/\delta y(x)$.

In spite of the fact that the domain of the functional $J[y]$ is infinite-dimensional, the line of reasoning that leads to the infinite-dimensional version of the Lagrange Multiplier Theorem 1 remains unchanged. In fact, the variational (i.e. "Euler") derivatives of $J[y]$ and of the $K_i[y]s$ correspond one for one to the familar gradient of $f(\vec{x}_0)$ and of the $g_i(\vec{x}_0)$s

$$\begin{array}{ccc} \dfrac{\delta J[y]}{\delta y(x)} & \leftrightarrow & \nabla f(\vec{x}_0) \\ \dfrac{\delta J K_1[y]}{\delta y(x)} & \leftrightarrow & \nabla g_1(\vec{x}_0) \\ \vdots & \leftrightarrow & \vdots \\ \dfrac{\delta J K_k[y]}{\delta y(x)} & \leftrightarrow & \nabla g_k(\vec{x}_0) \end{array} \tag{2.18}$$

The only difference between the first and the third colums is that the variational derivatives are on an infinite dimensional background, while the gradients are on its finite dimensional background.

The constrained optimization Theorem 2 below is the extension to an infinite dimensional background of the constrained optimization Theorem 1 for a finite dimensional background

Theorem 2.

Given: *Let*

$$J[y] = \int_{x_1}^{x_2} F(x, y, y')\, dx$$

be a scalar functional which has an optimal function $y(x)$ satisfying the k constraint conditions (2.6) on page 39.

Conclusion: *This optimal function satisfies the differeential equation*

$$F_y - \frac{d}{dx}F_{y'} - \lambda_1\left(G_{1y} - \frac{d}{dx}G_{1y'}\right) - \cdots - \lambda_k\left(G_{ky} - \frac{d}{dx}G_{ky'}\right) = 0 \ , \quad (2.19)$$

where the coefficients $\lambda_1, \cdots, \lambda_k$ *are obtained with the help of*

$$\int_{x_1}^{x_2} G_1(x, y, y')\, dx = c_1$$
$$\vdots$$
$$\int_{x_1}^{x_2} G_k(x, y, y')\, dx = c_k \ .$$

Example (Functional with two integral constraints):

Apply Lagrange's method to solve the variational problem

$$J[y] \equiv \int_{x_1}^{x_2} F(x, y, y')\, dx = \text{extremum}$$
$$K[y] \equiv \int_{x_1}^{x_2} G(x, y, y')\, dx = c_1 = \text{fixed}$$
$$L[y] \equiv \int_{x_1}^{x_2} H(x, y, y')\, dx = c_2 = \text{fixed}$$

This problem is a generalization to infinite dimensions of the example on page 45. One proceeds by first constructing

$$J^* = J - \lambda_1 K - \lambda_2 L$$

and then considering

$$\delta J^* = \delta(J - \lambda_1 K - \lambda_2 L) \ .$$

Thus

$$\delta J^* = \int \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\,\frac{\partial F}{\partial y'}\right)\delta f(x)dx - \lambda_1 \int \left(\frac{\partial G}{\partial y} - \frac{d}{dx}\,\frac{\partial G}{\partial y'}\right)\delta f(x)dx$$
$$-\lambda_2 \int \left(\frac{\partial H}{\partial y} - \frac{d}{dx}\,\frac{\partial H}{\partial y'}\right)\delta f(x)dx \ .$$

The problem is solved by setting

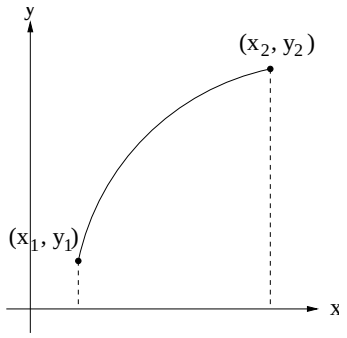
$$\delta J^* = \delta J - \lambda_1 \delta K - \lambda_2 \delta L = 0 \ .$$

Thus one has the following set of equations

$$\begin{aligned} \frac{\partial}{\partial y}(F - \lambda_1 G - \lambda_2 H) - \frac{d}{dx}\,\frac{\partial}{\partial y'}(F - \lambda_1 G - \lambda_2 H) &= 0 \\ \int_{x_1}^{x_2} G(x; y, y')dx &= c_1 \\ \int_{x_1}^{x_2} H(x; y, y')dx &= c_2 \end{aligned}$$

The solutions to these equations yield 4 constants:
(1) 2 integration constants
(2) 2 Lagrange multipliers
The 4 constants are determined by
(1) 2 boundary conditions on $y(x)$
(2) 2 constraints $K[y] = c_1$ and $L[y] = c_2$.

Example (Isoperimetric Problem):



Among all the curves $(x(\lambda),\, y(\lambda))$ of fixed length ℓ,

$$\int_{\lambda_1}^{\lambda_2} \sqrt{\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2}\, d\lambda = \text{constant} = \ell,$$

through the points (x_1, y_1) and (x_2, y_2) find that curve for which

$$\int_{\lambda_1}^{\lambda_2} y\, \frac{dx}{d\lambda}\, d\lambda = \text{extremum}$$

i.e. extremize the area!

Remark. Other formulations of this problem are

(i) $\int\ y\ dx$=extremum
subject to

$$\int \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \text{fixed}$$

(ii) $\int\ y\ \frac{dx}{ds}\ ds$=extremum
subject to

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$$

Solution: Consider the variational integrand

$$F^* = yx' - \mu\sqrt{x'^2 + y'^2}$$

where μ is a Lagrange multiplier. The Euler equations are

$$\frac{d}{d\lambda}\frac{\partial F^*}{\partial x'} = \frac{\partial F^*}{\partial x}$$

$$\frac{d}{d\lambda}\frac{\partial F^*}{\partial y'} = \frac{\partial F^*}{\partial y}$$

The fact that x is a cyclic coordinate, i.e. that $\frac{\partial F^*}{\partial x} = 0$ implies that $\frac{\partial F^*}{\partial x'}$ is constant:

$$y - \mu\frac{x'}{\sqrt{x'^2 + y'^2}} = \text{constant} \equiv C$$

The second equation gives us

$$\frac{d}{d\lambda}\left((-)\mu\frac{y'}{\sqrt{x'^2 + y'^2}}\right) - x' = 0$$

The form of these equations suggests that one introduce arclength,

$$ds = \sqrt{x'^2 + y'^2}d\lambda,$$

as a new curve parameter. In terms of this parameter we have

$$\frac{1}{\sqrt{\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2}}\ \frac{d}{d\lambda} = \frac{d}{ds} = \text{“ . ”}$$

We also notice that for *any* curve extremal or nonextremal, the use of s as a curve parameter implies

$$\sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2} = 1 \tag{2.20}$$

In terms of the derivative with respect to arclength the Euler equations become

$$y - \mu\dot{x} = c \tag{2.21}$$

and

$$\mu\ddot{y} + \dot{x} = 0 \tag{2.22}$$

Comments about consistency: Any two of the equations (2.20), (2.21), and (2.22) imply the third. Thus, notice that the last two equations are consistent with the requirement that

$$\boxed{\dot{x}^2 + \dot{y}^2 = \text{const} = 1.} \tag{2.23}$$

This requirement was already highlighted on page 37, but we also validate it by direct differentiation: Differentiate Eq(2.21)

$$\mu\ddot{x} - \dot{y} = 0 \tag{2.24}$$
$$\mu\ddot{y} + \dot{x} = 0 \tag{2.25}$$

These are the unintegrated Euler equations. Insert them into the derivative of $\dot{x}^2 + \dot{y}^2$,

$$\frac{d}{ds}(\dot{x}^2 + \dot{y}^2) = 2\ddot{x}\dot{x} + 2\ddot{y}\dot{y} = 2\ddot{x}(-)\lambda\ddot{y} + 2\ddot{y}\lambda\ddot{x} = 0 \tag{2.26}$$

Thus $\dot{x}^2+\dot{y}^2$=constant along the curve. This is consistent with Eq.(2.20). This constancy is a general property and we shall return to it when we discuss the symmetries of the functional J.

Now return to the task of solving Eqs (2.21) and (2.22). Insert Eq.(2.21) into (2.22) and obtain

$$\ddot{y} + \frac{1}{\mu^2}\, y = \frac{c}{\mu^2}$$

The solution is

$$\boxed{y = A\cos\left(\frac{s}{\mu} + \delta\right) + c} \tag{2.27}$$

To obtain $x(s)$, use Eq.(2.21),

$$\dot{x} = \frac{y - c}{\mu} = \frac{A}{\mu}\cos\left(\frac{s}{\mu} + \delta\right) \ ,$$

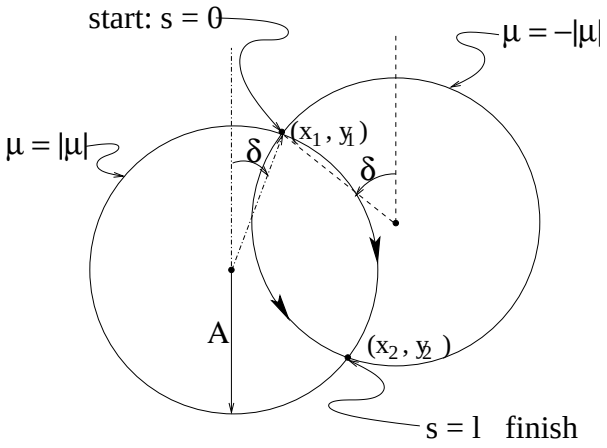


Figure 2.4: Two intersecting circles as solutions to the isoperimetric problem.

and obtain

$$\boxed{x = A\sin\left(\frac{s}{\mu} + \delta\right) + x_0} \tag{2.28}$$

Equations (2.27) and (2.28) are parametrized equations for a *circular arc* because $(y-c)^2 + (x-x_0)^2 = A^2$.

We have five constants A, δ, μ, c, and x_0 whose values are determined by five conditions, namely,

$$\text{At } s = 0: \quad x(0) = x_1 \tag{2.29}$$

$$y(0) = y_1 \tag{2.30}$$

$$\text{At } s = \ell: \quad x(\ell) = x_2 \tag{2.31}$$

$$y(\ell) = y_2 \tag{2.32}$$

$$\int_0^\ell \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2}\, ds = \ell\ . \tag{2.33}$$

This integral constraint together with Eq.(2.26) imply that that

$$\dot{x}^2 + \dot{y}^2 = 1,$$

which is to say that the squared length of the tangent to the curve has unit length. Q: What does this equation tell us about the Lagrange multiplier μ? A: With the help of the solutions $x(t)$ and $y(t)$ it implies that

$$\frac{A^2}{\mu^2} = 1\ . \tag{2.34}$$

It follws that $\mu = \pm A$. Thus there are *two* solutions, one corresponding to $\mu = A$, which gives an arc whose length increases with angle in a clockwise fashion; the other corresponds to $\mu = -A$, which gives an arc whose length increases with angle in a counter-clockwise fashion as the curve proceeds from point 1 to point 2.
The other implication is also geometrical: for both circles the Langrange multiplier $|\mu|$ is the radius of their circular arcs.

Economy and Transparency

Mathematical economy and geometrical transparency are always cognitive virtues. In the above isoperimetric problem they are achieved by the method of complex arithmetic: Consolidate the two real degrees of freedom into a single complex degree of freedom,

$$z(s) = y(s) + ix(s),$$

and its complex conjugate,

$$\overline{z}(s) = y(s) - ix(s),$$

This collapses the two Euler Eqs.(2.24)-(2.25) into a single equation

$$\mu\ddot{z} - i\dot{z} = 0$$

whose solution is

$$\begin{aligned} z(s) &= Ce^{i\frac{s}{\mu}} \\ &= Ae^{i(\frac{s}{\mu}+\delta)} \qquad (2.35) \\ &= \underbrace{A\cos\left(\frac{s}{\mu}+\delta\right)}_{y(s)} + i\underbrace{A\sin\left(\frac{s}{\mu}+\delta\right)}_{x(s)} \qquad (2.36) \end{aligned}$$

The Lagrange multiplier μ is real. Consequently, the complex conjugate equation $\mu\ddot{\bar{z}} + i\dot{\bar{z}} = 0$ yields merely the complex conjugate of Eq.(2.36). Furthermore, the constraint Eq.(2.23) when expressed in terms of the complex solution, Eq.(2.35) and its complex conjugate, namely

$$\dot{z}\dot{\bar{z}} = 1,$$

leads to the real result Eq.(2.34) on page 52, as it must.

Lecture 6

2.2.3 Variational Problems with Finite Constraints

Aside from isoperimetric variation problems, i.e. those having a set of integral constraints as subsidiary conditions, another important class is exemplified by the following:
Find the curve $\{y(x), z(x)\}$ through points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ such that

$$J = \int F(x, y, z, y', z')dx = \text{extremum}$$

subject to the "finite" constraint

$$G(x, y, z) = 0 \ .$$

Given that this problem has a solution, we consider the first variation of J due to deviations $\delta y(x), \delta z(x)$ of the variants $y + \delta y$ and $z + \delta z$ away from the functions $y(x)$ and $z(x)$:

$$\begin{aligned}\delta J \ &= \ \int_{x_1}^{x_2} \left[\left(F_y - \frac{d}{dx} F_{y'} \right) \delta y(x) + \left(F_z - \frac{d}{dx} F_{z'} \right) \delta z(x) \right] dx \\ &\quad + F_{y'} \delta y \Big|_{x_1}^{x_2} + F_{z'} \delta z \Big|_{x_1}^{x_2}\end{aligned}$$

The fact that the variant $\{y(x) + \delta y(x), \ z(x) + \delta z(x)\}$ is required to lie on the constraint manifold $G = 0$, i.e., it must satisfy

$$G(x, y + \delta y, z + \delta z) = 0 \ ,$$

implies that the variations δy and δz are not independent. They must satisfy

$$\frac{\partial G}{\partial y} \delta y(x) + \frac{\partial G}{\partial z} \delta z(x) = 0$$

so that

$$\delta z = -\frac{G_y}{G_z} \delta y.$$

Insert this into the expression for the first variation and obtain

$$\delta J = \int_{x_1}^{x_2} \left\{ (F_y - \frac{d}{dx} F_{y'}) - (F_z - \frac{d}{dx} F_{z'}) \frac{G_y}{G_z} \right\} \delta y dx + F_{y'} \delta y \Big|_{x_1}^{x_2} - F_{z'} \frac{G_y}{G_z} \delta y \Big|_{x_1}^{x_2}$$

The endpoint terms are zero because all variants pass through the same pair of end points (x_1, y_1, z_1) and (x_2, y_2, z_2) so that $\delta y(x_1) = \delta y(x_2) = 0$.
The variations

$$\delta y = \sum \alpha_i h_i(x)$$

are arbitrary. Consequently, $\delta J = 0$ implies that for each x one has

$$\frac{F_y - \frac{d}{dx} F_{y'}}{G_y} = \frac{F_z - \frac{d}{dx} F_{z'}}{G_z}$$

Let us designate this common function by $\lambda(x)$. Thus one obtains two Euler equations,

$$\boxed{F_y - \frac{d}{dx} F_{y'} - \lambda(x) G_y = 0 \text{ and } F_z - \frac{d}{dx} F_{z'} - \lambda(x) G_z = 0}.$$

These equations together with

$$\boxed{G(x, y(x), z(x)) = 0}$$

determine the three functions

$$y(x), z(x), \text{ and } \lambda(x).$$

Even though $\lambda = \lambda(x)$ not a constant, it still is referred to a "Lagrange multiplier."

2.3 Variable End Point Problem

It is not necessary that the endpoints of a variational problem be fixed. Consider as an example the Brachistochrone problem (Lecture 3) in which the second end point, instead of being fixed, is required to lie somewhere on the vertical line. Which of the curves will allow a sliding bead to reach its destination in the least amount of time?

The general problem is this: Find that curve $y(x)$, which starts at $x = a$ and ends at $x = b$, for which

$$J = \int_a^b F(x, y, y') dx + \psi(y(b)) - \varphi(y(a)) = \text{ extremum}, \tag{2.37}$$

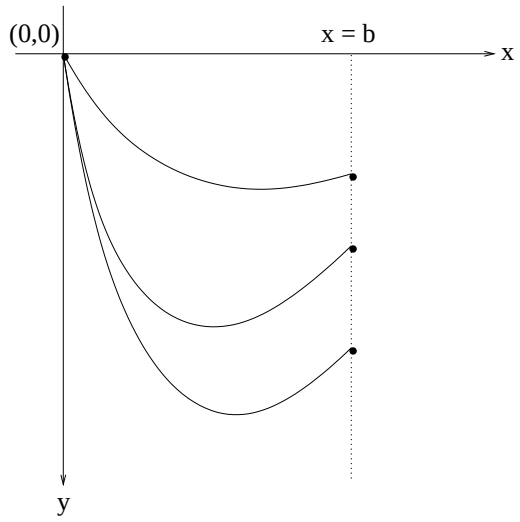


Figure 2.5: Brachistochrone trial curves with different end points.

where F, ψ, and φ are given.

The variational principles we have considered so far can be said to have had *Dirichelet* boundary conditions. By contrast, the present variational principle can be said to have just the opposite type, namely "natural" or "free" boundary conditions.

Having intergrated by parts, one finds that the first variation of the functional J is

$$\delta J = \int_a^b [F_y - \frac{d}{dx} F_{y'}] \delta f(x) dx + \left(\frac{\partial F}{\partial y'} + \frac{\partial \psi}{\partial y} \right) \delta f(x) \Big|^{x=b} - \left(\frac{\partial F}{\partial y'} + \frac{\partial \varphi}{\partial y} \right) \delta f(x) \Big|^{x=a}$$

The variations δf fall into two main classes:

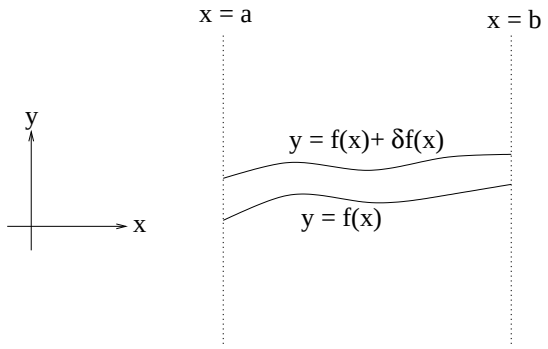


Figure 2.6: Trial curves with variable starting points and end points.

class 1: $\delta f : \delta f(a) = 0 \;\; \delta f(b) = 0$

class 2: $\delta f : \delta f(a) \neq 0 \;\; \delta f(b) \neq 0$

We know that $J =$ extremum $\Rightarrow \delta J = 0$ for all variations δf .
First we let δf be of class 1. In that case there are no endpoint terms in δJ. Consequently, $\delta J = 0 \Rightarrow F_y - \frac{d}{dx}(F_{y'}) = 0$.
Thus we are left with

$$\delta J = \left(F_{y'} + \frac{\partial \psi}{\partial y}\right)\delta f(x)\Big|^{x=b} - \left(F_{y'} + \frac{\partial \varphi}{\partial y}\right)\delta f(x)\Big|^{x=a}.$$

Next we let δf be of class 2. Then

$$\delta J = 0 \Rightarrow \left(F_{y'} + \frac{\partial \psi}{\partial y}\right)\Big|^{x=b} = 0 \text{ and } \left(F_{y'} + \frac{\partial \varphi}{\partial y}\right)\Big|^{x=a} = 0.$$

These are the optimal boundary conditions associated for the given variational principle, Eq.(2.37), with variable endpoints as expressed by the boundary terms ψ and ϕ.

2.3.1 Extremum Principle at a Moment of Time Symmetry

The extremum principle with side conditions and with variable end points is illustrated by the following example.

Consider a vibrating string imbedded in an elastic medium and with a spring attached to each end. The tension of the string, the coefficient of restitution of the elastic medium, and the spring constants are properties which determine the temporal evolution, i.e. the vibrations, of the system. These properties we assume to be independent of time. Consequently, the system is said to be time-invariant. One can show that for such a time-invariant system its total energy,

$$\text{T.E.} = \text{Kinetic energy} + \text{Potential energy}$$

is independent of time, i.e. the total energy of a time invariant system is conserved.

In general the vibratory motion of the system is such that its kinetic energy is never zero. Moreover, there exist certain types of motion, namely those in which the vibrations are characterized by a single frequency. Such states of motions are called *normal modes*. They have the property that at some instant of time the system is momentarily stationary. At that moment the system's velocity is zero, its kinetic energy is zero and all of its energy is in the form of potential energy,

$$\text{T.E.} = 0 + \text{P.E. (at a moment of time symmetry)}$$

We inquire as to the amplitude profile of the system at this moment of time symmetry. The amplitude profile which nature allows is governed by a extremum principle.

Consider the total energy of the system at a moment of time symmetry ($\frac{\partial y}{\partial t} = 0$)

$$\begin{aligned} \text{T.E.} = J[y] &= \frac{1}{2}\int_a^b \left[T(x)\left(\frac{dy}{dx}\right)^2 + \kappa(x)y^2\right] dx \\ &+ \frac{1}{2}k_1\left(y(a) - y_1\right)^2 + \frac{1}{2}k_2\left(y(b) - y_2\right)^2. \end{aligned}$$

This is the potential energy of the system. It consists of three parts:

1. The "stretch" energy

$$\int_a^b T\sqrt{(dx)^2 + (dy)^2} - \int_a^b T dx \approx \frac{1}{2}\int_a^b T\left(\frac{dy}{dx}\right)^2 dx$$

   Here $T = T(x)$ is the (position dependent) string tension.

2. The displacement energy due to the string being displaced in an elastic medium with a local restoration ("Hook's") coefficient

$$\kappa(x)dx$$

   for each incremental interval dx.

3. The potential energy stored in the two springs at $x = a$ and $x = b$. Their spring constants are k_1 and k_2.

The extremum principle governing the amplitude profile at a moment of time symmetry is given by

$$J[y] = \text{extremum} \tag{2.38}$$

subject to

$$K[y] = \frac{1}{2}\int_a^b \rho(x)y^2 dx = 1. \tag{2.39}$$

Here $\rho(x)$ is the linear mass density of the string.
With the help of the Lagrange multiplier λ, this principle reads

$$J[y] - \lambda K[y] = \text{extremum}.$$

Nota bene. It turns out that this principle is equivalent to

$$\frac{J[y]}{K[y]} = \text{extremum}.$$

This is Rayleigh's variational principle and we shall consider it later.

The Euler equation for either variational principle yield the same differential equation, namely

$$-\frac{d}{dx}T(x)\frac{dy}{dx} + (K(x) - \lambda\rho(x))y = 0 \tag{2.40}$$

This is the result of what we called "class 1" variations on page 56.

By contrast,"class 2" variations, which do not vanish at $x = a$ yield

$$T(x)y'(x)\Big|^a + k_1(y(b) - y_1) = 0$$

or

$$\frac{T(a)}{k_1}y'(a) + y(a) = y_1 \tag{2.41}$$

Similarly at $x = b$ one has

$$\frac{T(b)}{k_2}y'(b) + y(b) = y_2 \tag{2.42}$$

Equations 2.41 and 2.42 are known as the inhomogeneous mixed Dirichelet Neumann boundary conditions for the vibrating string. If we let the springs become infinitely stiff, i.e. let $k_1 \to \infty$ and $k_2 \to \infty$, then the problem reduces to a Dirichelet boundary value problem where the end point amplitude has the fixed values

$$y(a) = y_1$$
$$y(b) = y_2$$

We conclude that the Sturm-Liouville boundary value problem, Equations 2.40, 2.41 and 2.42 is a consequence of the variational principle,

$$\text{P.E.} - \lambda\int_a^b \frac{1}{2}\rho(x)y^2\,dx = \text{extremum.} \tag{2.43}$$

Remark 1. In vibrational problems the Lagrange multiplier λ, the eigenvalue of the S-L problem, refers to the squared vibrational angular frequency,

$$\lambda \equiv \omega^2 = (2\pi \text{ frequency})^2,$$

of the system, and the kinetic energy (K.E.) of the system vibrating at this frequency is

$$\frac{1}{2}\omega^2\int_a^b \rho(x)y^2dx.$$

Consequently, the amplitude profile of such a normal mode is governed by the variational principle

$$\text{P.E.} - \text{K.E.} = \text{extremum.}$$

We shall see later that the "least action principle"

$$\int_{t_1}^{t_2} (\text{P.E. - K.E.})\, dt = \text{extremum}$$

yields not only the amplitude profile at any instant of time, but also the evolution of this profile with time. Evidently, equation (2.43), the least energy problem with its amplitude constraint is a specialization of the least action principle.
Remark 2. This principle can be extended to two and three dimensions where the vibration can be those of a drum, solid, liquid sphere (vibrating sphere, vibrating neutron star, vibrating uranium nucleus), vibrating black hole, sound in cavity, vibrating bridge, etc.

Lecture 7

2.4 Generic Variable Endpoint Problem

There are variable end point problems in which the endpoint lies on a curve or a surface. For example,

1. what is the optimal curve from a point to a surface, or for that matter from one surface to another?

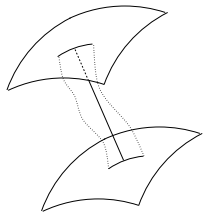


Figure 2.7: Optimal curve and its variants between two surfaces.

2. How does an extremal meet a terminating surface?

3. Given that a functional extends over two different media,

$$J = \int_a^c F(x, y, y')\, dx \; + \; \int_c^b F^*(x, y, y')\, dx \;,$$

how must an extremal in $x < c$ be related to to its continuation in $c < x$, i.e. what are the junction conditions that prevail at the interface. For example, what path does light take as it passes from one medium to another?

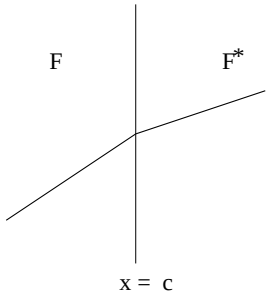


Figure 2.8: Curve crossing the boundary between two media characterized by F and F^*.

These questions and others can be answered within the framework of the following generic variable end point problem:
Find the curve $\{y_1(x), \ldots, y_n(x)\}$ in an $n+1$ dimension space for which the variational integral

$$J[y_1, \ldots, y_n] = \int_{x_0}^{x_1} F(x, y, \ldots, y_n, y_n', \ldots, y_n')dx = \text{extremum}$$

subject to the constraint that the starting and end points $\{y_i(x_0)\}$ and $\{y_i(x_1)\}$ lie on two prescribed curves (or surfaces).
This variational principle asks us to consider changes in J not only due to variation in the curves,

$$y_i \to y_i + \delta y_i,$$

but also due to variations in the points

$$\begin{aligned} x_0 &\to x_0 + \delta x_0 \\ x_1 &\to x_1 + \delta x_0, \end{aligned}$$

where the curves start and finish. The determination of these changes in J is a task which one must do before the endpoint constraints are imposed.
To simplify discussion we consider the simplest case

$$J = \int_{x_0}^{x_1} F(x, y, y')dx = \text{extremum}$$

and calculate the first variation δJ allowing the endpoints to vary. Consider the curves

		starts	ends
curve:	$y(x)$	$P_0 = (x_0, y_0)$	$P_1 = (x_1, y_1)$
its variant:	$y^*(x)$	$P_0^* = (x_0 + \delta x_0, y_0 + \delta y_0)$	$P_1^* = (x_1 + \delta x_1, y_1 + \delta y_1)$

$$\boxed{h(x) = y^*(x) - y(x) = \text{difference at fixed } x}$$

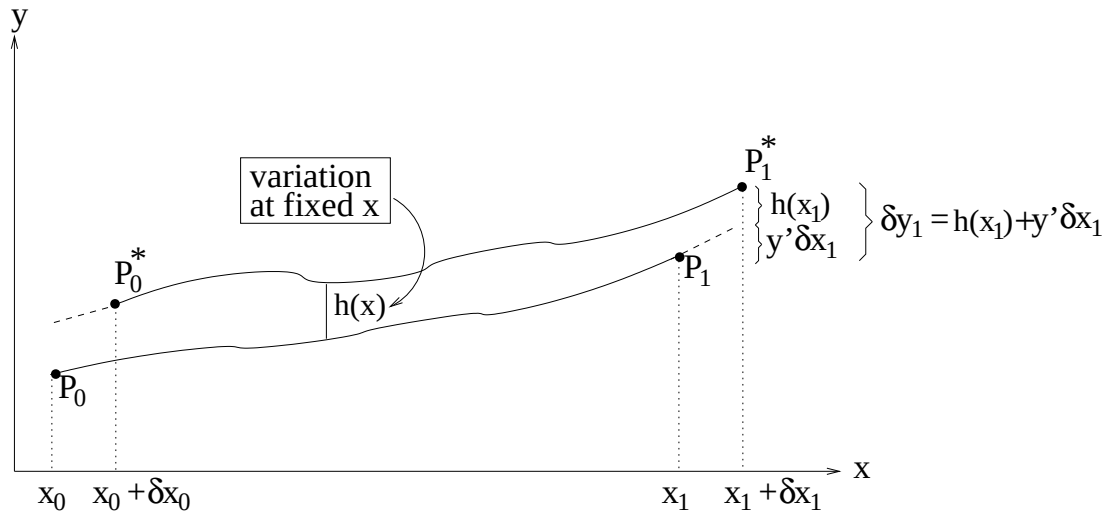


Figure 2.9: A curve and its generic variant.The total vertical variation δy_1 includes only the principal linear part. With that stipulation one has $h(x_1) = h(x_1 + \delta x_1)$.

2.4.1 General Variations in the Functional

The total variation in the functional to be extremized is

$$\begin{aligned}
\Delta J &= \int_{x_0+\delta x_0}^{x_1+\delta x_1} F(x, y+h, y'+h')dx - \int_{x_0}^{x_1} F(x, y, y')dx \\
&= \int_{x_0}^{x_1} [F(x, y+h, y'+h') - F(x, y, y')]dx \\
&+ \int_{x_0+\delta x_0}^{x_0} F(x, y+h, y'+h')dx + \int_{x_1}^{x_1+\delta x_1} F(x, y+h, y'+h')dx
\end{aligned}$$

whose principal linear part is

$$\Delta J = \int_{x_0}^{x_1} [F_y - \frac{d}{dx}F_{y'}]h(x)dx + F_{y'}h(x)\Big|_{x_0}^{x_1} + F(x, y, y')\delta x\Big|_{x_0}^{x_1}.$$

The last term is new. It is proportional to the shift in x :

$$\delta x\Big|^{x_0} = \delta x_0 \quad \text{and} \quad \delta x\Big|^{x_1} = \delta x_1.$$

Referring to Figure 2.9, we express any type of end point variation in terms of the independent variations δy_i and δx_i :

$$\begin{aligned}
h(x_0) &= \delta y_0 - y'(x_0)\delta x_0 \\
h(x_1) &= \delta y_1 - y'(x_1)\delta x_1
\end{aligned}$$

Thus one obtains for the total first variation

$$\begin{aligned}\Delta J &= \int_{x_0}^{x_1} \frac{\delta J}{\delta y(x)} h(x)dx + F_{y'}\delta y\Big|_{x=x_0}^{x=x_1} \\ &\quad +(F - y'\frac{\partial F}{\partial y'})\delta x\Big|_{x=x_0}^{x=x_1} \end{aligned} \tag{2.44}$$

We have two endpoint terms because in general both y and x are allowed to change at the boundary.
This formula opens the door to a new mathematical perspective in optimizing the integral $J[y]$. It is the means

(a) to mathematize boundary conditions

(b) to mathematize junction conditions

(c) of entering into the "canonical" formulation of mechanics

(d) of entering into the Hamilton-Jacobi formulation of mechanics.

As we shall see in Chapter 3.5, the conceptual economy of this Hamilton-Jacobi formulation is breath-taking. This is because a system of several ordinary differential equations, the E-L Eqs.(2.1) on page 34, gets replaced by a *single* partial differential equation, the H-J equation.

In order to get already now a general idea of the relevant line of reasoning, consider the context of two close-by optimal curves $y(x)$ and $y^*(x)$, both satisfying the same E-L equation

$$0 = \frac{\delta J}{\delta y(x)} \equiv F_y - \frac{d}{dx}F_{y'} \ , \tag{2.45}$$

and each of them passing through its pairs of points

$$P_0 = (x_0, y_0) \qquad \& \qquad P_1 = (x_1, y_1) \tag{2.46}$$

and through

$$P_0^* = (x_0 + \delta x_0, y_0 + \delta y_0) \qquad \& \qquad P_1^* = (x_1 + \delta x_1, y_1 + \delta y_1) \ . \tag{2.47}$$

The expression for ΔJ, Eq.(2.44), becomes

$$\Delta J = F_{y'}\delta y\Big|_{x=x_0}^{x=x_1} + (F - y'\frac{\partial F}{\partial y'})\delta x\Big|_{x=x_0}^{x=x_1} \ .$$

It implies that J is a well-defined function of its end points,

$$J = J[y](x_0, y_0; x_1, y_1),$$

whenever $y(x)$ is an optimal curve. Because of this one has the following two partial derivatives

$$\frac{\partial J}{\partial x}\Big)_y \equiv \lim_{\delta x_1 \to 0} \frac{\Delta J}{\delta x_1}\Big)_y = F(x; y, y') - y' F_{y'}(x; y, y') \tag{2.48}$$

$$\frac{\partial J}{\partial y}\Big)_x \equiv \lim_{\delta y_1 \to 0} \frac{\Delta J}{\delta y_1}\Big)_x = F_{y'}(x; y, y') \tag{2.49}$$

Solving Eq.(2.49) for y' in terms of $\frac{\partial J}{\partial y}$ one obtains the partial differential equation

$$\frac{\partial J}{\partial x} + H\left(x; y, \frac{\partial J}{\partial y}\right) = 0 \ .$$

Its solution is the scalar function

$$J = J(x, y) \ .$$

Its determination replaces the task of solving the Euler-Lagrange differential Eq.(2.45), and hence the task of exhibiting the optimal extremals $y(x)$ in closed form. The physical significance, the mathematical nature of this scalar as well as the method by which it yields the extremal curves comprize the heart of the new mathematical perspective. We shall develop these in Chapter 3.5 after the Lagrangian and Hamiltonian mathematization of dynamics.

2.4.2 Transversality Conditions

Continuing the mathematization of optimized junction and boundary conditions, we now subject the first variation to prescribed endpoint constraints. We require that the paths start and terminate on the two respective given curves

$$y = \varphi(x) \tag{2.50}$$

and

$$y = \psi(x) \tag{2.51}$$

or equivalently,

$$x = \Phi(y) \tag{2.52}$$

and

$$x = \Psi(y) \tag{2.53}$$

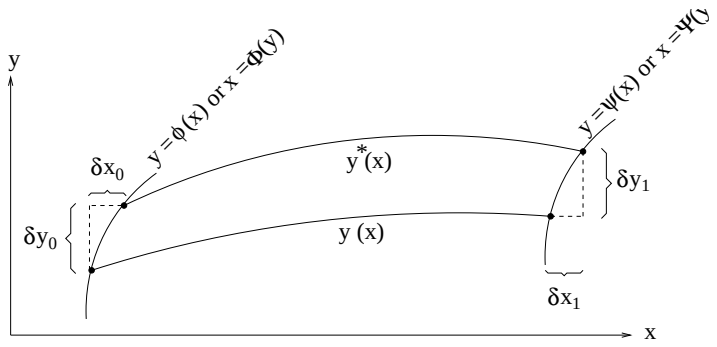


Figure 2.10: A curve and its variant with end point constraints.

In that case the vertical and horizontal variations in the end point are related by

$$\delta y_0 = \varphi'(x_0)\delta x_0 \quad \text{and} \quad \delta y_1 = \psi'(x_1)\delta x_1 \tag{2.54}$$

Consequently, the first variation in J is

$$\begin{aligned} \Delta J &= \int_{x_0}^{x_1} \frac{\delta J}{\delta y} h(x)dx + [F_{y'}\psi'(x) + (F - y'F_{y'})]_{x=x_1}\delta x_1 \\ &\quad -[F_{y'}\varphi'(x) + (F - y'F_{y'})]_{x=x_0}\delta x_0 \end{aligned}$$

If $y(x)$ is an optimal curve, then $\Delta J = 0$ for arbitrary variations $h(x)$ and arbitrary variations δx_i at the end points. The former implies that $y(x)$ satisfies the Euler equation. The latter implies

$$\begin{aligned} \Big[F + (\psi' - y')F_{y'}\Big]_{x=x_1} &= 0 \\ \Big[F + (\varphi' - y')F_{y'}\Big]_{x=x_0} &= 0, \end{aligned}$$

which are known as the *delta x transversality conditions*. By contrast, the *delta y transverality conditions* are

$$\begin{aligned} \Big[F_{y'} + (F - y'F_{y'})\,\Psi'(y)\Big]_{(x,y)=(x_1,y_1)} &= 0 \\ \Big[F_{y'} + (F - y'F_{y'})\,\Phi'(y)\Big]_{(x,y)=(x_0,y_0)} &= 0 \end{aligned}$$

One must use this kind of transversality condition if there are many degrees of freedom, $\{\, y_1(x), y_2(x), \cdots, y_n(x)\,\}$, as is the case in Section 2.5 on page 68.

2.4.3 Junction Conditions

Suppose one wished to find an optimal curve between two points in adjoining regions characterized by different variational integrands. Thus we consider an optimal curve which leaves one region characterized by F and enters another characterized by F^* so that

$$J = \int_a^c F(x, y, y')dx + \int_c^b F^*(x, y, y')dx = \text{extremum}.$$

The boundary between the two adjoining regions is $x = c.$, and it is depicted in Figure 2.8 on page 61. The extremum principle establishes a relationship between the properties of the curve in the two regions. To determine this relation we, as usual, consider the total first variation

$$\begin{aligned}\Delta J &= \int_a^c \frac{\delta J}{\delta y}\delta y(x)dx + \int_c^b \frac{\delta J}{\delta y}\delta y(x)dx \\ &\quad + F_{y'}\delta y\Big|_a^{c^-} + F^*_{y'}\delta y\Big|_{c^+}^b \\ &\quad + (F - y'\frac{\partial F}{\partial y'})\delta x\Big|_a^{c^-} + (F^* - y'\frac{\partial F^*}{\partial y'})\delta x\Big|_{c^+}^b .\end{aligned}$$

The fact that the curve is optimal implies

$$\frac{\delta J}{\delta y} = 0.$$

The fact that the boundary between the two regions is $x = c$, implies that $\delta x(c) = 0$, i.e. it is fixed and given.
Consequently the extremal condition implies

$$0 = \frac{\Delta J}{\delta y(c)} = F_{y'}\Big|_{c^+}^{c^-}$$

$$\boxed{F_{y'}\Big|_{c^+} = F^*_{y'}\Big|_{c^-}} \text{ The Junction Condition}$$

Example 1. *(Snell's Law).*

Consider a light beam propagating from one medium where the refractive index is $n(x, y)$ to another where it is $n^*(x, y)$. The variational principle is

$$J = \int_a^c n(x, y)\sqrt{1 + y'^2}dx + \int_c^b n^*(x, y)\sqrt{1 + y'^2}dx = \text{extremum!}$$

The junction condition is

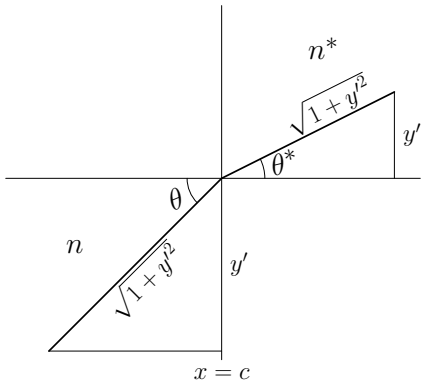


Figure 2.11: Ray refraction at the interface between two media having constant refractive indeces n and n^*.

$$n(x,y)\frac{y'}{\sqrt{1+y'^2}}\bigg|_{x=c^-} = n^*(x,y)\frac{y'}{\sqrt{1+y'^2}}\bigg|_{x=c^+},$$

which according to Figure 2.11 is geometrized by the statement

$$n\sin\theta = n^*\sin\theta^*.$$

This is Snell's law of refraction.

Example 2. *(Segmented string)*

Consider a string with discontinuous tension. Such a discontinuity would prevail at the junction of two sting segments $0 \le x < c$ and $c < x \le \ell$ which are joined at $x = c$ by a frictionless ring. This ring (having neglegible mass) is constrained to slide without friction along a rod transverse to the segmented string. However, at the same time this rod pulls with non-zero force along the longitudinal direction of that string. Consequently, there is a corresponding non-zero jump in the tension of the string segments along their junction. In fact, the difference, $T_2(c^+) - T_1(c^-)$, in the tension

$$T(x) = \begin{cases} T_1(x) & x < c \\ T_2(x) & c < x \end{cases}$$

on the two sides of the junction $x = c$ equals the longitudinal force provided by the frictionless rod.
The equation for the amplitude profile of a normal mode of any string is

$$\frac{d}{dx}T(x)\frac{dy}{dx} + [q(x) - \lambda\rho(x)]\,y = 0 \tag{2.55}$$

Q What are the junction conditions across the frictionless support rod?

A The variational principle for the amplitude profile is

$$\text{P.E.} = \frac{1}{2}\int_0^c \left[T_1\, y'^2 + q\, y^2\right] dx + \frac{1}{2}\int_c^\ell \left[T_2\, y'^2 + q\, y^2\right] dx = \text{extremum} \quad (2.56)$$

subject to

$$\int_0^\ell \rho\, y^2 dx = 1. \quad (2.57)$$

The Euler equationfor this "isoperimetric" problem is given by Eq.(2.55), where λ is the Lagrange multiplier for the given constraint, Eq.(2.57). The junction condition

$$F_{y'}|_{x=c^+} = F_{y'}|_{x=c^-}$$

yields

$$T_2(x)\left.\frac{dy}{dx}\right|_{c^+} = T_1(x)\left.\frac{dy}{dx}\right|_{c^-}.$$

In other words, the *transverse* force on the string parallel to the boundary $x = c$ is continuous across the boundary.

2.5 Many Degrees of Freedom

One would like to generalize the variable end point problem from extremal paths in the plane to extremal path in a higher dimensional space. In that case the two curves (2.52) and (2.53) generalize to the two surfaces

$$x = \Phi(y_1, \ldots, y_n) \quad (2.58)$$

and

$$x = \Psi(y_1, \ldots, y_n) \quad (2.59)$$

for the loci of points where the extremal path starts and finishes.

The corresponding generalization for the total first variation consists of replacing the equation (2.44) with

$$\begin{aligned}\Delta J &= \int_{x_0}^{x_1} \sum_{i=1}^{n} \frac{\delta J}{\delta y_i(x)} \delta y_i(x) dx + \sum_{i=1}^{n} F_{y_i'} \delta y_i \Big|_{x_0}^{x_1} \\ &\quad + \left(F - \sum_{j=1}^{n} y_j' \frac{\partial F}{y_j'}\right) \delta x \Big|_{x_0}^{x_1}\end{aligned}$$

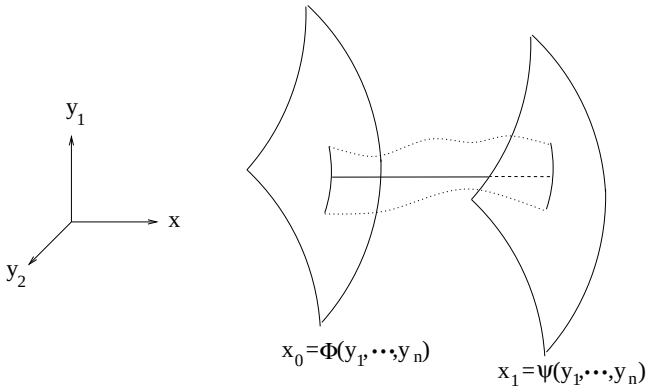


Figure 2.12: A curve and its variant between surfaces.

Conceptual unit economy demands that it is best not to single out one of the y_i's over all the others. Consequently, we express the variation in δx at in terms of the variations of all the δy_i's in the surfaces (2.58) and (2.59),

$$\delta x_0 = \sum_{i=1}^{n} \frac{\partial \Phi}{\partial y_i}\, \delta y_i$$
$$\delta x_1 = \sum_{i=1}^{n} \frac{\partial \Psi}{\partial y_i}\, \delta y_i.$$

With the help of these expressions the transversality conditions at these starting and terminating surface are given by

$$\left[F_{y_i'} + \frac{\partial \Phi}{\partial y_i} \left(F - \sum_{j=1}^{n} y_j' \frac{\partial F}{\partial y_j'} \right) \right]_{x=x_0} = 0, \qquad i = 1, \ldots, n$$

and analogous condition at $x = x_1$, where the gradient of the surface is $\left\{ \frac{\partial \Psi}{\partial y_i} \right\}_{i=1}^{n}$.

Example 3. *(Fermat's Principle of Least Time).*
Consider Fermat's principle of least time for a light beam propagating through a medium whose refractive index is $n(x, y, z)$. Let the light beam start at (x_0, y_0, z_0) and finish at some point x_1, y_1, z_1 on the surface

$$x = \Psi(y, z)$$

such that the path minimizes the time of travel. The quantity to be extemized is

$$\begin{aligned} J[y, z] &= \frac{1}{c} \int_0^1 n(x, y, z) \sqrt{dx^2 + dy^2 + dz^2} \\ &= \frac{1}{c} \int_{x_0}^{x_1} n(x, y, z) \sqrt{1 + y'^2 + z'^2}\, dx \end{aligned}$$

One applies the transversality conditions to the variational integrand. We have

$$\begin{aligned} F &= n(x,y,z)\sqrt{1+y'^2+z'^2} \\ \frac{\partial F}{\partial y'} &= \frac{ny'}{\sqrt{1+y'^2+z'^2}}; \qquad \frac{\partial F}{\partial z'} = \frac{nz'}{\sqrt{1+y'^2+z'^2}} \\ F - y'F_{y'} - z'F_{z'} &= n\sqrt{1+y'^2+z'^2} - \frac{y'^2+z'^2}{\sqrt{1+y'^2+z'^2}} \\ &= \frac{n}{\sqrt{1+y'^2+z'^2}} \end{aligned}$$

There are two transversality conditions at $x = x_1$. They are

$$\frac{ny'}{\sqrt{1+y'^2+z'^2}} + \frac{\partial \Psi}{\partial y}\frac{n}{\sqrt{1+y'^2+z'^2}} = 0$$

and

$$\frac{nz'}{\sqrt{1+y'^2+z'^2}} + \frac{\partial \Psi}{\partial z}\frac{n}{\sqrt{1+y'^2+z'^2}} = 0 \ .$$

Thus after cancellation we obtain

$$(y', z') = (-\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial z})$$

or

$$(x', y', z') = (1, -\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial z}) = \overrightarrow{\nabla} g$$

The left hand side is $\frac{d\vec{x}}{dx}$, the tangent to the light beam. the right hand side is the gradient of the function $g(x,y,z) = x - \Psi(y,z)$. This gradient is $\perp$ to the surface. Thus the transversality condition says that *the optimal light beam strikes the surface perpendicularly.*

2.6 Parametrization Invariant Problem

A mathematical problem is said to be geometrical if it is a statement whose validity does not depend on the coordinate system used in its enunciation.

Thus, the shortest trajectory between two point is a straight line, or the Pythagorean theorem, are geometrical statements.

In geometrical problems where one is dealing with the determination of curves which are optimal, it is more suitable not to single out any one coordinate over the others as an independent variable. Instead, represent these curves parametrically. Thus let

$$x = x(\lambda)$$

$$y = y(\lambda)$$

The choice of curve parameterization is optional, i.e. different parametrizations represent the same curve provided the functions x(λ) and y(λ) do *not* satisfy

$$\frac{dx}{d\lambda} = 0 \qquad \text{and} \qquad \frac{dy}{d\lambda} = 0$$

simultaneously. This resttriction guarantees that λ expresses non-zero progress along the curve.

2.6.1 Parametrization Invariance via Homogeneous Function

Consider the simplest variational problem. Its variational integral has the form

$$J = \int_{x_0}^{x_1} F\left(x, y, \frac{dy}{dx}\right) dx = \int_{\lambda_0}^{\lambda_1} G\left(x, y, \frac{dx}{d\lambda}, \frac{dy}{d\lambda}\right) d\lambda$$

where

$$G = \frac{dx}{d\lambda} F\left(x, y, \frac{\frac{dy}{d\lambda}}{\frac{dx}{d\lambda}}\right)$$

This integrand is a function which is "homogeneous" in the variables $\frac{dx}{d\lambda}$ and $\frac{dy}{d\lambda}$ and its "degree of homogeneity" is 1. It satisfies the homogeneity relation

$$G\left(x, y, k\frac{dx}{d\lambda}, k\frac{dy}{d\lambda}\right) = kG\left(x, y, \frac{dx}{d\lambda}, \frac{dy}{d\lambda}\right) \tag{2.60}$$

Following Euler we differentiate this expression with respect to k, and then settig $k = 1$, we obtain

$$x'G_{x'} + y'G_{y'} = G, \text{ where} x' = \frac{dx}{d\lambda} \text{ and} y' = \frac{dy}{d\lambda}. \tag{2.61}$$

Conversely, suppose that G is some homogeneous function of degree one in x' and y', i.e G satisfies Eq.(2.60), then the variational principle

$$\int_{\lambda_0}^{\lambda_1} G d\lambda = extremum$$

determines a parametrization invariant path. This is so becuase the parameter change $\lambda = \lambda(\overline{\lambda})$ yields

$$\begin{aligned}
\int_{\overline{\lambda}_0}^{\overline{\lambda}_1} G\left(x, y, \frac{dx}{d\overline{\lambda}}, \frac{dy}{d\overline{\lambda}}\right) d\overline{\lambda} &= \int_{\overline{\lambda}_0}^{\overline{\lambda}_1} G\left(x, y, x'\frac{dx}{d\overline{\lambda}}, y'\frac{dy}{d\overline{\lambda}}\right) d\overline{\lambda} \\
&= \int_{\overline{\lambda}_0}^{\overline{\lambda}_1} G\left(x, y, x', y'\right) \frac{d\lambda}{d\overline{\lambda}} d\overline{\lambda} \\
&= \int_{\lambda_0}^{\lambda_1} G(x, y, x', y') d\lambda
\end{aligned}$$

We conclude that *the variational problem is invariant under a change in parametrization*, and that the extremal path is independent of one's choice of parametrization used to represent it. The extremal path satisfies the Euler equations

$$G_x - \frac{d}{d\lambda} G_{x'} = 0 \qquad \text{and} \qquad G_y - \frac{d}{d\lambda} G_{y'} = 0. \tag{2.62}$$

These equations must be equivalent to the single equation

$$F_y - \frac{d}{dx} \frac{\partial F}{\partial \left(\frac{dy}{dx}\right)} = 0$$

This means that somehow the two equations (2.62) cannot be independent equations. There must exist an identity between them. In fact, one can show with the help of Eq. (2.61) that

$$x'(G_x - \frac{d}{d\lambda} G_{x'}) + y'(G_y - \frac{d}{d\lambda} G_{y'}) = 0$$

is an identity. It holds for an any path, extremal or non-extreamal.

We shall now apply these ideas to the geometrical problem of finding the geodesics in a space coordinatized by curvilinear coordinates.

2.7 Variational Principle for a Geodesic

Consider flat Euclidean space coordinatized by (x,y,z) = $\vec{x}$. Introduce curvilinear coordinates (x^1, x^2, x^3) as an alternative means of labelling the points

$$\left.\begin{array}{l} x(x^1, x^2, x^3) \\ y(x^1, x^2, x^3) \\ z(x^1, x^2, x^3) \end{array}\right\} \vec{r}(x^i).$$

We would like to reexpress distance

$$(ds)^2 = dx^2 + dy^2 + dz^2$$

in terms of these curvilinear coordinates. Thus we consider

$$dx = \sum_{i=1}^{3} \frac{\partial x}{\partial x^i} dx^i$$

$$dy = \sum_{i=1}^{3} \frac{\partial y}{\partial x^i} dx^i$$

$$dz = \sum_{i=1}^{3} \frac{\partial z}{\partial x^i} dx^i$$

The distance formula becomes

$$\begin{aligned}(ds)^2 &= \sum_i \sum_j dx^i \frac{\partial x}{\partial x^i}\frac{\partial x}{\partial x^j}dx^j + dx^i \frac{\partial y}{\partial x^i}\frac{\partial y}{\partial x^j}dx^j + dx^i \frac{\partial z}{\partial x^i}\frac{\partial z}{\partial x^j}dx^j \\ &= \sum_i \sum_j dx^i \frac{\partial \vec{r}}{\partial x^i}\cdot\frac{\partial \vec{r}}{\partial x^j}dx^j \\ &= g_{ij}(x^k)dx^i dx^j\end{aligned}$$

Remark. The last line intoduces the *Einstein summation convention* in which a *pair of repeated indeces* implies a summation over the relevent coordinates. Also note that the coefficients

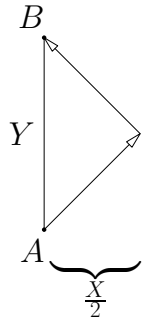
$$g_{ij} = \frac{\partial \vec{r}}{\partial x^i}\cdot\frac{\partial \vec{r}}{\partial x^j} = \vec{e}_i \cdot \vec{e}_j \tag{2.63}$$

are the inner products between ith and jth coordinate tangent vectors. They would not be mutually orthonormal in an oblique curvilinear coordinate system.

Q: How does one determine the optimal path between two given points A and B?

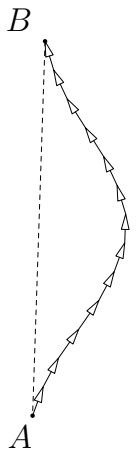
A: The optimal path is the one which extremizes the path length between A and B.

The simplest case is a broken curve consisting of two parts:



$$\begin{aligned}distance &= 2\sqrt{\left(\frac{Y}{2}\right)^2 + \left(\frac{X}{2}\right)^2} = \sqrt{Y^2+X^2} \\ &= \text{extremum for the direct } (X{=}0) \text{ path}\end{aligned}$$

For broken curves consisting of several pieces, we have



$$\sum_i \sqrt{\Delta x_i^2 + \Delta y_i^2} = extremum.$$

Among arbitrary curves beween A and B, the path length between A and B

$$\int_A^B \left[g_{ij} dx^i dx^j \right]^{1/2} = \begin{pmatrix} \text{minimum for a straightline} \\ \text{as compared to} \\ \text{any of its variants} \end{pmatrix} ,$$

i.e. *extremal length* is an indicator of straightness.

Remark. The introduction of curvilinear coordinates in Euclidean space is optional. In such a space one can always find a global coordinate system x,y,z such that

$$(ds)^2 = dx^2 + dy^2 + dz^2.$$

In other spaces, e.g. a curved surface or more generally a Rimannian space, there is no option. The non-Euclidean nature of the space makes the introduction of curvilinear coordiantes mandatory. Relative to such coordinates one necessarily has

$$(ds)^2 = g_{11} \left(dx^1 \right)^2 + 2 g_{12} dx^1 dx^2 + g_{22} \left(dx^2 \right)^2 + \cdots$$

A particularly exotic example is the interior of a star. In terms of spherical coordinates (r,θ,ϕ), the distance formula is not the expected expression $(ds)^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$, but instead is given by

$$(ds)^2 = \frac{dr^2}{1 - \frac{8\pi}{3}\frac{G}{c^2}\rho r^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$

where the constants ρ, G, and c are

$$\begin{aligned} \rho &= \frac{\text{mass}}{\text{volume}} \\ G &= \text{Newton's gravitational constant } \left(= \frac{1}{15,\,000,\;000} \text{ in cgs units} \right) \\ c &= \text{speed of light } (= 3 \times 10^{10} \text{ in cgs units}) \end{aligned}$$

This expression is a consequence of Einstein's law of graviation.

Introducing the spatial scale parameter

$$a = \left(\frac{8\pi}{3} \frac{G}{c^2} \rho \right)^{-1/2}, \tag{2.64}$$

the distance formula has the form

$$(ds)^2 = \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right).$$

This suggests that one let

$$r = a \sin \chi$$

for the purpose of simplifying the distance formula. Doing so results in

$$(ds)^2 = a^2 \left(d\chi^2 + \sin^2 \chi \left(d\theta^2 + \sin^2 \chi d\phi^2 \right) \right).$$

One can show that this is the element of distance on part of a three-sphere S^3

$$x^2 + y^2 + z^2 + w^2 = a^2$$

of radius a, and which is coordinatized by

$$\begin{aligned} x &= a \ \sin \chi \sin \theta \cos \phi \\ y &= a \ \sin \chi \sin \theta \sin \phi \\ z &= a \ \sin \chi \cos \theta \\ w &= a \ \cos \chi \quad . \end{aligned}$$

The remarkable feature of this three-sphere is that its radius a is determined by the stars mass density[2] in Eq.(2.64).

[2]For example, with a density of 1 gram/cc, that "radius" would be ∼60 solar radii.

2.8 Equation of Geodesic Motion

To determine the curve having extremal distance between A and B, we pick a curve with the suspected property and deform it slightly, but arbitrarily

$$\begin{aligned} \text{original curve: } x^i &= a^i(\lambda) \\ \text{variant = deformed curve: } x^i &= a^i(\lambda) + \delta a^i(\lambda)\,. \end{aligned}$$

1. Along either curve the distance is

$$s = \int_A^B ds = \int_0^1 \left[g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} \right]^{1/2} d\lambda\,. \tag{2.65}$$

2. At *fixed* λ we find that

   (a) the metric coefficient $g_{ij}\left(x^b(\lambda)\right)$ differs from one cuve to the other by

$$\delta g_{ij} \equiv g_{ij}\left(a^k(\lambda) + \delta a^k(\lambda)\right) - g_{ij}\left(a^k(\lambda)\right) = \frac{\partial g_{ij}}{\partial x^k} \delta a^k(\lambda)\,,$$

   (b) the components $\frac{dx^i}{d\lambda}$ of the tangent vector differ by

$$\delta\left(\frac{dx^i}{d\lambda}\right) \equiv \frac{d(a^i + \delta a^i)}{d\lambda} - \frac{da^i}{d\lambda} = \frac{d}{d\lambda}(\delta a^i)\,,$$

   (c) the application of these changes in g_{ij} and $\frac{dx^i}{d\lambda}$ to the integrand of Eq.(2.65) results in the following change in the integrated distance, Eq.(2.65),

$$\delta s = \int_0^1 \frac{\frac{1}{2} g_{ij} \frac{d}{d\lambda}(\delta a^i) \frac{da^j}{d\lambda} + \frac{1}{2} g_{ij} \frac{da^i}{d\lambda} \frac{d}{d\lambda}(\delta a^j) + \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} \delta a^k \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}}{\left[g_{mn} \frac{da^m}{d\lambda} \frac{da^n}{d\lambda} \right]^{1/2}} d\lambda\,,$$

   The first two term are equal. Integrate them by parts and omit the boundary term because both paths must pass through A and B:

$$\delta a^i(0) = \delta a^i(1) = 0\,.$$

   Thus obtain

$$\delta s = \int_0^1 f_k(\lambda) \delta a^k \left[g_{mn} \frac{da^m}{d\lambda} \frac{da^n}{d\lambda} \right]^{1/2} d\lambda\,,$$

   where

$$f_k(\lambda) = \frac{-1}{\left[g_{mn} \frac{da^m}{d\lambda} \frac{da^n}{d\lambda} \right]^{1/2}} \frac{d}{d\lambda} \left(\frac{g_{kj} \frac{da^j}{d\lambda}}{\left[g_{mn} \frac{da^m}{d\lambda} \frac{da^n}{d\lambda} \right]^{1/2}} \right) + \frac{\frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}}{\left[g_{mn} \frac{da^m}{d\lambda} \frac{da^n}{d\lambda} \right]}\,. \tag{2.66}$$

An extremum is achieved, and the first order change δs vanishes for every first order deformation δa^k of the path $a^k(\lambda)$, when the three quantities that multiply the δa^k all vanish. Thus one obtains the n conditions

$$f_k(\lambda) = 0 \qquad k = 1, 2, 3, \cdots, n$$

for the determination of an optimal path.

In 3-dimensional Euclidean or Riemannian space these are three equations. In 4-dimensional Lorentz space-time these are four equations, even though in this latter case the space-time interval is given by

$$(d\tau)^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \,,$$

provided gravitation is absent.

Lecture 8

Lecture 8

2.9 Geodesics: Their Parametrization.

The equations $f_k(\lambda) = 0$ constitute overkill! Why?

2.9.1 Parametrization Invariance.

It is because their number is more than enough to express the extrmal nature of

$$s = \int_0^1 [g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}]^{\frac{1}{2}} d\lambda$$

This follows from the parametrization independence of this integral. The reparametrization

$$\begin{aligned} \lambda \to \bar{\lambda} &= \lambda + h(\lambda) \\ \bar{\lambda}(0) &= 0 \\ \bar{\lambda}(1) &= 1 \Rightarrow h(0) = h(1) = 0 \end{aligned}$$

does not change the integral. It corresponds to a mere "repositioning of beads along a string" (= reparametrization)

$$s = \int_0^1 [g_{ij} \frac{dx^i}{d\bar{\lambda}} \frac{dx^j}{d\bar{\lambda}}]^{\frac{1}{2}} d\bar{\lambda}$$

The change in $a^i(\lambda)$ brought about by such a reparametrization is

$$a^i(\lambda) \to a^i(\lambda + h(\lambda)) = a^i(\lambda) + \delta a^i(\lambda)$$

where

$$\delta a^i(\lambda) = \frac{da^i}{d\lambda} h(\lambda)$$

The fact that such variations cannot change the variational integral for arbitrary reparametrization magnitudes $h(\lambda)$ implies that

$$\boxed{f_k(\lambda)\frac{da^k)}{d\lambda} = 0} \tag{2.67}$$

The remarkable feature of this equation is that it holds for *any* path $a^k(\lambda)$, even if it is not optimal.

An equation which holds true whether or not the quantities obey any differential equation is called an *identity.*

Thus instead of needing n independent equations, one needs only $n-1$ equations to specify an extremal path (in n dimensions).

2.9.2 Parametrization in Terms of Curve Length

This reparametrization freedom can be exploited to introduce a physically fundamental parameter, the length parameter itself,

$$ds = \left[g_{mn} \frac{da^m}{d\lambda} \frac{da^n}{d\lambda} \right]^{\frac{1}{2}} d\lambda$$

Let $x^i = a^i(\lambda)$ be the optimal path. With s as the new path parameter call this extrmal path $x^i(s)$. Making this replacement in the differential equation

$$f_k(\lambda) = 0$$

one obtains

$$0 = g_{kj} \frac{d^2x^j}{ds^2} + \frac{\partial g_{kj}}{\partial x^m} \frac{dx^m}{ds} \frac{dx^j}{ds} - \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} \frac{dx^i}{ds} \frac{dx^j}{ds}$$

The middle term can be rewritten as

$$\frac{1}{2} \frac{\partial g_{kj}}{\partial x^m} \dot{x}^m \dot{x}^j + \frac{1}{2} \frac{\partial g_k j}{\partial x^m} \dot{x}^j \dot{x}^m$$

Let $m = \bar{j}$, $j = \bar{m}$ and then drop the bar. The result of this "index gymnastics" is

$$0 = g_{kj} \frac{d^2x^j}{ds^2} + \frac{1}{2} \left(\frac{\partial g_{kj}}{\partial x^m} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{mj}}{\partial x^k} \right) \frac{dx^m}{ds} \frac{dx^j}{ds} \equiv f_k \tag{2.68}$$

Introduce the inverse g^{lk} of the matrix g_{kj}:

$$g^{lk}g_{kj} = \delta^l_j$$

The equation of the geodesic becomes

$$\boxed{\frac{d^2x^l}{ds^2} + \Gamma^l_{\ mj}\frac{dx^m}{ds}\frac{dx^j}{ds} = 0} \tag{2.69}$$

where

$$\Gamma^l_{\ mj} = \frac{1}{2}g^{lk}\left(\frac{\partial g_{kj}}{\partial x^m} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{mj}}{\partial x^k}\right)$$

is the so called "Christoffel symbol of the 2^{nd} kind".

Remark. By contrast, without the $g^{\ell k}$, the coefficients

$$\Gamma_{k\ mj} \equiv \frac{1}{2}\left(\frac{\partial g_{kj}}{\partial x^m} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{mj}}{\partial x^k}\right)$$

are the so called the "Christoffel symbols of the 1^{nd} kind". As we shall see later, both kinds characterize the differential geometric law of parallel transport. Both kinds have at first sight a daunting appearance, but they are computationally very powerful in obtaining this law from one's knowledge of the inner products $g_{ij} = \vec{e}_i \cdot \vec{e}_j$ at each point.

Comment. The identitiy Eq.(2.67) on page 78, or equivalently,

$$f_k\frac{dx^k}{ds} = 0 \tag{2.70}$$

should not come as a surprise. Recall that the Pythagorean theorem, $(\Delta s)^2 = g_{ij}\Delta x^i \Delta x^j$, implies

$$g_{kj}\frac{dx^k}{ds}\frac{dx^j}{ds} = 1, \tag{2.71}$$

even for nongeodesic curves; and, because of the arclength parametrization, the tangent $\left\{\frac{dx^k}{ds}\right\}$ has been normalized to unit length. It therefore is an integral of motion of the system of differential Eqs.(2.68) or (2.69). Indeed, differentiating the l.h.s. of Eq.(2.71) with respect to s yields

$$\frac{d}{ds}\left(g_{kj}\frac{dx^k}{ds}\frac{dx^j}{ds}\right) = f_k\frac{dx^k}{ds}, \tag{2.72}$$

where f_k is given by the r.h.s. of Eq.(2.68). The result is therefore this:
If $f_k = 0$ or if Eq.(2.69) holds, then $g_{kj}\frac{dx^k}{ds}\frac{dx^j}{ds}$ is a constant, an integral of motion. Conversely, given *any* s-parametrized curve along which $g_{kj}\frac{dx^k}{ds}\frac{dx^j}{ds}$ is constant, then Eq.(2.70) must be satisfied, *even* if $f_k \neq 0$.

2.10 Physical Significance of the Equation for a Geodesic

Q What is the physical significance and purpose of the Γ-coefficients in the equation for a geodesic?

A These coefficients express the acceleration of a free particles relative to a given frame of reference. There are three archetypical reference frames: (i) free float coordinate frames, e.g. an orbiting space capsule not rotating relative to the fixed stars, (ii) rotating coordinate frames, and (iii) linearly accelerated coordinate frames.

2.10.1 Free float frame

Also called "inertial" frames of reference, such frames are defined by Newton's 1st Law of Motion:
This means that relative to a free float frame free particle move uniformly along straight lines, i.e. these particles obey Newton's 1st law of motion. Mathematically we have

$$\text{mass} \times \text{acceleration}{=}0 \;\Rightarrow \frac{dx^i}{ds} = (const)^i \Leftrightarrow (\text{straight line motion})$$

where the travelled distance is

$$s = (constant) \times \text{time} \;\Leftrightarrow (\text{uniform motion})$$

2.10.2 Rotating Frame

Consider a frame of reference which rotates with angular velocity $\vec{\omega}$ relative to the fixed stars. First, consider a vector, say $\vec{G}$, which is rigidly attached to this rotating frame. The vector will rotate relative to the static inertial frame, which is static relative to the stars. In fact, during a time interval dt this rotating vector $\vec{G}$ will have changed by an amount which is given by

$$(d\vec{G})_{static} = dt\,\vec{\omega} \times \vec{G}$$

This expresses an infinitesimal rotation around the axis $\vec{\omega}$. Second, consider the circumstance in which the vector $\vec{G}$ is not rigidly attached to the rotating frame. Instead, let it change by the amount $(d\vec{G})_{rot}$ during the time interval dt, then the total change in this vector will be

$$(d\vec{G})_{static} = (d\vec{G})_{rot} + dt\,\vec{\omega} \times \vec{G}$$

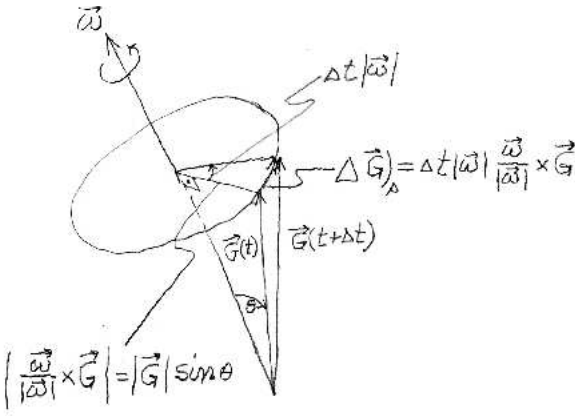


Figure 2.13: Rigid frame rotating relative to the fixed stars with angular velocity $\vec{\omega}$. Its direction as well as its magnitude $|\vec{\omega}|$ are taken to be fixed relative to the fixed stars. Being fixed relative to the rotasting frame, the vector $\vec{G}(t)$ rotates relative to the fixed stars with angular velocity $\vec{\omega}$. The vector $\frac{\vec{\omega}}{|\vec{\omega}|}\times\vec{G}$ is $\perp$ to $\vec{\omega}$ and $\vec{G}$ but its length $\left|\frac{\vec{\omega}}{|\vec{\omega}|}\times\vec{G}\right|$ is equal to that of the radius vector $\perp$ to $\vec{\omega}$.

relative to the inertial frame static with respect to the fixed stars. Thus

$$\frac{d}{dt}\Big)_{static} = \frac{d}{dt}\Big)_{rot} + \vec{\omega}\times$$

Finally, apply this transformation law to the postion vector $\vec{R}$,

$$\frac{d\vec{R}}{dt}\Big)_{static} = \frac{d\vec{R}}{dt}\Big)_{rot} + \vec{\omega}\times\vec{R}$$

and then to the velocity vector $\frac{d\vec{R}}{dt}\Big)_{static} \equiv \vec{v}_s$. Assuming that $\vec{\omega}$ is constant, one obtains that the equation of motion for a free particle is

$$0 \;=\; m\frac{d^2\vec{R}}{dt^2}\Big)_{static} = m\left[\vec{a}_{rot} + 2\vec{\omega}\times\vec{v}_{rot} + \vec{\omega}\times(\vec{\omega}\times\vec{R})\right]$$

with

$$\vec{v}_{rot} = \frac{d\vec{R}}{dt}\Big)_{rot}$$

and

$$\vec{a}_{rot} = \frac{d\vec{v}_{rot}}{dt}\Big)_{rot}$$

In terms of components one has

$$m\frac{d^2x^i}{dt^2}\Big)_{rot} = \underbrace{-2m[\vec{\omega}\times\vec{v}_{rot}]^i}_{Coriolis\,force}\underbrace{-m[\vec{\omega}\times(\vec{\omega}\times\vec{R})]^i}_{centrifugal\,force} \qquad (8.8) \qquad (2.73)$$

relative to the *rotating* coordinate frame. The two terms on the right hand side of this equation are the Coriolis force" and the "centrifugal force" in these frames. Suppose we try to compare these Newtonian equations of motion with those of a geodesic,

$$\frac{d^2x^i}{ds^2} = -\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}$$

This is a good thing to do because both sets of equations have seond derivatives on the left hand side. The obstacles are the independent variables, time t for Newton's equations, and geodesic length s for the geodesic equations. But this difference is no problem because for each free particle path, length and time are linearly related:

$$\begin{aligned} s &= (constant)\times \text{time} \\ &\equiv (constant)\,t \end{aligned} \qquad (2.74)$$

Consequently, the geodesic equations are equivalent to

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{jk}\frac{dx^i}{dt}\frac{dx^k}{dt} \qquad (8.10) \qquad (2.75)$$

$$\frac{d^2t}{dt^2} = 0\,. \qquad (8.11) \qquad (2.76)$$

Suppose we introduce time

$$x^0 = t\,.$$

as another coordinate so that

$$\frac{dx^0}{dt} = 1$$

and

$$\frac{d^2x^0}{dt^2} = 0\,. \qquad (2.77)$$

We now consolidate the two equations (2.75) and (2.77) into the single four-component equation

$$\frac{d^2x^\mu}{dt^2} = -\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} \qquad \mu = 0,1,2,3 \qquad (2.78)$$

$$= -\Gamma^\mu_{00}\frac{dx^0}{dt}\frac{dx^0}{dt} - \sum_{k=1}^{3}(\Gamma^\mu_{0k}+\Gamma^\mu_{k0})\frac{dx^0}{dt}\frac{dx^k}{dt} - \sum_{j=1}^{3}\sum_{k=1}^{3}\Gamma^\mu_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} \qquad (2.79)$$

The fact that $\frac{dx^0}{dt} = 1$ results in

$$\frac{d^2x^\mu}{dt^2} = -\Gamma^\mu_{00} - \sum_{k=1}^{3}(\Gamma^\mu_{0k} + \Gamma^\mu_{k0})\frac{dx^k}{dt} - \sum_{j=1}^{3}\sum_{k=1}^{3}\Gamma^\mu_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}$$

What are these Γ-coefficients? They are determined by doing a component-by-component comparison between this four-component equation with the three-plus-one set of Eqs.(2.73) and (2.77) and requiring that these two sets are equivalent for all particles. For example, the first component ($i = 1$) of Eq.(2.73) is

$$\frac{d^2x^1}{dt^2} = -2(\omega^2\frac{dx^3}{dt} - \omega^2\frac{dx^2}{dt}) - (\vec{\omega}\cdot\vec{\omega}x^1 - \vec{\omega}\cdot\vec{R}\omega^2)$$

For $\mu = 0$ equivalence implies

$$\Gamma^0_{\alpha\beta} = 0 \qquad \alpha, \beta = 0, 1, 2, 3\ ,$$

while for $\mu = i\ (= 1, 2, 3)$ one finds that in the rotating frame

$$\Gamma^i_{0k}\frac{dx^k}{dt} = \left(\vec{\omega}\times\frac{d\vec{R}}{dt}\right)^i \qquad i = 1, 2, 3 \quad \text{(Coriolis acceleration)}$$

so that[3]

$$\begin{matrix} i=1\to \\ i=2\to \\ i=3\to \end{matrix} \left[\quad \Gamma^i_{0k} \quad\right] \begin{bmatrix} dx^1/dt \\ dx^2/dt \\ dx^3/dt \end{bmatrix} = \begin{bmatrix} 0 & \omega^3 & -\omega^2 \\ -\omega^3 & 0 & \omega^1 \\ \omega^2 & -\omega^1 & 0 \end{bmatrix} \begin{bmatrix} dx^1/dt \\ dx^2/dt \\ dx^3/dt \end{bmatrix}\ .$$

Furthermore,

$$\Gamma^i_{00} = \left[\vec{\omega}\times(\vec{\omega}\times\vec{R})\right]^i \qquad i = 1, 2, 3 \quad \text{(centrifugal acceleration)}\,.$$

In other words,

$$\begin{aligned} \Gamma^i_{00} &\neq 0 \Leftrightarrow \text{centrifugal force} \neq 0 \\ \Gamma^i_{0k} &\neq 0 \Leftrightarrow \text{Coriolis force} \neq 0 \end{aligned}$$

This says that the "Christoffel symbols" of the equations for a geodesic are an expression of centrifugal and Coriolis accelerations, which are associated with a rotating coordinate frame. Let us now extend these same considerations to an accelerated frame of reference.

[3]The coefficients Γ^μ_{0k} and of Γ^μ_{k0} of the quadratic form in the geodesic equation occur only in the symmetric combination $(\Gamma^\mu_{0k} + \Gamma^\mu_{k0})$. Consequently, one may assume without loss of generality that $\Gamma^\mu_{0k} = \Gamma^\mu_{k0}$.

2.10.3 Uniformly Accelerated Frame

Recall that relative to a free float ("inertial") frame the equation for the free particle moving along the x-direction is

$$\frac{d^2x}{dt^2} = 0\,. \tag{2.80}$$

Consider the motion of such a particle relative to an accelerated frame.

Q What is the equation of motion of this particle relative to a frame accelerating into the positive x-direction with acceleration g?

A To find out, consider a meter rod parallel to the x-direction, but attached to the accelerated frame.

Let ξ be the coaccelerating coordinate displacement along this meter rod. Thus a coaccelerating point on this meter stick is measured to be ξ units from the floor.

However, relative to the nonaccelerated inertial frame the x-coordinate of this point on the meter stick is

$$x = \xi + \frac{1}{2}g\tau^2$$

with

$$t = \tau$$

These two equations comprise the coordinate transformation $(\tau, \xi) \rightarrow (t, x)$ between the two frames. Let

$$\xi = \xi(\tau)$$

be the particle trajectory as measured in the accelerated frame. Appling the coordinate transformation to Newton's Eq.(2.80) for the free particle yields

$$0 = \frac{d^2}{d\tau^2}[\xi(\tau) + \frac{1}{2}g\tau^2] = \frac{d^2\xi}{d\tau^2} + g\,.$$

Furthermore,

$$0 = \frac{d^2t}{d\tau^2}\,.$$

Let $x^0 = t$ and $x^1 = \xi$. The equations assume the form

$$\begin{aligned} \frac{d^2x^0}{d\tau^2} &= 0 \\ \frac{d^2x^1}{d\tau^2} &= -g \end{aligned}$$

Compare them to the equation for a geodesic in two dimensions:

$$\begin{aligned}\frac{d^2x^0}{d\tau^2} &= -\Gamma^0_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \\ \frac{d^2x^1}{d\tau^2} &= -\Gamma^1_{00}\left(\frac{dx^0}{d\tau}\right)^2 - 2(\Gamma^1_{01}+\Gamma^1_{10})\frac{dx^0}{d\tau}\frac{dx^1}{d\tau} - \Gamma^1_{11}\left(\frac{dx^1}{d\tau}\right)^2\end{aligned}$$

Assume they apply to all possible particle trajectories, we find

$$\Gamma^1_{00} = g = \text{"inertial acceleration"} \qquad (8.15) \tag{2.81}$$

All the other Γ's are zero. One concludes therefore that relative to a uniformly accelerated frame

$$\Gamma^1_{00} \neq 0 \Leftrightarrow \text{"non-inertial force"} \neq 0\,.$$

2.11 The Equivalence Principle and "Gravitation"="Geometry"

Einstein directed attention to the fact that there is no way that one can tell the difference between (i) the motion of free particles in a uniformly accelerated frame, and (ii) the motion of particle falling freely in a uniform gravitational field.

This holds true regardless of the particle's composition. In other words, the particle trajectories are identical, no matter whether a particle is made of gold, aluminum, or snakewood. The measurements were first done by Lorand von *Eötvös*. Thus the inertial force is equvalent to, i.e. physically 'indistinguishable from, the gravitational force.

This equivalence is called *the equivalence principle*. It implies that

$$\text{"inertial force"} = \text{"gravitational force"}\,, \tag{2.82}$$

or

$$\text{mass}_{inertial} \times \text{"inertial acceleration"} = \text{mass}_{gravitational} \times \text{gravitational field}$$

This equation holds regarless of whether the inertial mass and the gravitational mass refer to particles made of gold, aluminum, etc. Thus

$$\frac{m_{in}}{m_{acc}} = 1\,,$$

which expresses the fact that this ratio is independent of the composition of the paricles. Doing the cancellation and appying Eq.(2.81) to Eq. (2.82), we find that the Christoffel symbol Γ^1_{00} is to be identified with the graviational field, i.e.

$$\Gamma^i_{00} = (\text{"gravitational field"})^i \quad i = 1,2,3 \qquad (8.18a) \tag{2.83}$$

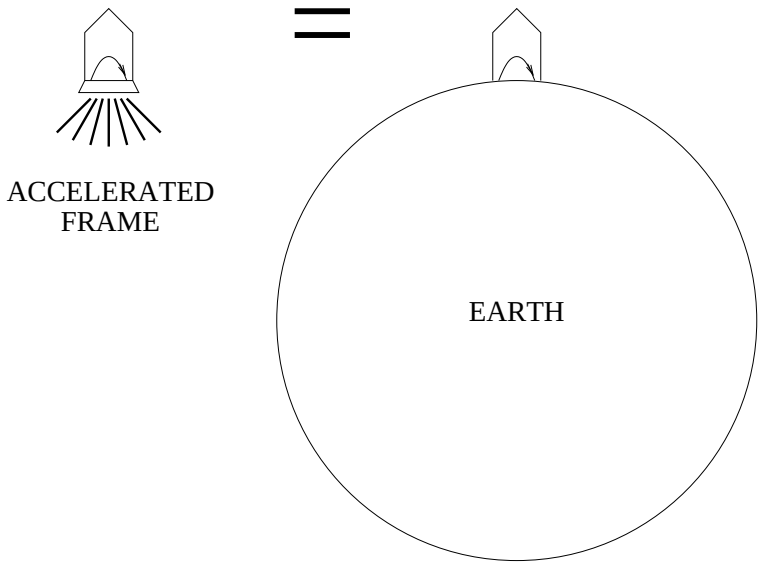


Figure 2.14: Equivalence Principle: the trajectories of particles in an accelerated frame and in a uniform gravitational field are indistinguishable regardless of their composition.

if we had extended our considerations to motion in three dimenssions.

Apply this result to a static gravitational field. In that case

$$\Gamma^i_{00} = \sum_{\alpha=0}^{3} \frac{1}{2} g^{i\alpha}(g_{\alpha 0,0} + g_{\alpha 0,0} - g_{00,\alpha})$$

The fact that the gravitatioal field is independent of time means that all time-derivatives must vanish. Consequently, the expression reduces to

$$\Gamma^i_{00} = \sum_{j=1}^{3} \frac{1}{2} g^{ij}(-) g_{00,i} \tag{2.84}$$

If the gravitational field is weak, then we can say that distance measurements in three dimenssional space are governed by Euclidean geometry as expressed by the theorem of Pythagoras,

$$(ds)^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \,.$$

This implies that the matrix, Eq. (2.63), on page 73 has the simple form

$$[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Consequently, its inverse $[g^{ij}]$is also the identity matrix,

$$[g^{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Applying this to the simplified Christoffell symbol, Eq.(2.84), give the result

$$\Gamma^i_{00} = -\frac{1}{2}\frac{\partial g_{00}}{\partial x^i} \qquad i = 1, 2, 3$$

Introduce this simplified result into Eq. (2.83) and obtain

$$-\frac{1}{2}\frac{\partial g_{00}}{\partial x^i} = (\text{gravitational field})^i \qquad i = 1, 2, 3 .$$

Recall that the gravitational field is the negative gradient of Newton's gravitational potential $\Phi(x^1, x^2, x^3)$,

$$(\text{gravitational field})^i = -\frac{\partial}{\partial x^i}\Phi .$$

Consequently,

$$\Phi(x^1, x^2, x^3) = \frac{1}{2} g_{00}(x^1, x^2, x^3) .$$

This is a new result. It says that

$$\text{"gavitational potential"} = \text{"metric coefficient"} .$$

In other words, by examing the motion of particles we find that gravitation manifests itself by altering the properties of space (and time) as expressed by the coefficients in the "metric"

$$(ds)^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

and by the concomitant Christoffel symbols

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma}(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma}) \qquad \mu, \alpha, \beta = 0, 1, 2, 3$$

One summarizes this result by saying that Einstein's *equivalence principle* leads to the conclusion that

$$\text{"gravitation"} = \text{"geometry"} .$$

Remark. The reasoning process that leads from (a) the equivalence between (i) trajectories of free particles in an accelerated frame and (ii) the trajectories of particles in a gravitaqtional field to (b) the identification of gravitation as a geometrical property is an example of the scientific method, induction based on experiments and/or observation.

Chapter 3

Variational Formulation of Mechanics

3.1 Hamilton's Principle

Lecture 9

Recall Newton's second law of motion applied to a force derived from a potential $U(x, y, z, t)$ is

$$\frac{d}{dt}\left(m\dot{\vec{x}}\right) = -\vec{\nabla}U(x, y, z, t)\,.$$

Like all differential equations, this law expresses – in mathematical form – a causal relation in the world. Here the change in momentum is caused by a force. This law governs the change in the particle's momentum along every point of the particle's trajectory. This law is therefore a local one, it is a differential statement about the motion of the particle. By integrating this differential equation one obtains the global trajectory, an integral curve. This is another causal relation, this time between the particle's initial position and velocity on one hand and its final position and velocity on the other. If one knows the initial velocity at point event A, then the integration of Newton's equation yields a unique curve which terminates at, say, point event B.

One now asks: given that there exists a unique curve through A and B, does there exist a principle which singles out this curve over any other – a principle with a mathematical starting point different from Newton's local equations of motion?

The answer is given by the principle of least (more generally, stationary) action, also called *Hamilton's Principle*:

> Of all possible paths along which a dynamical system may (be imagined to) move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.

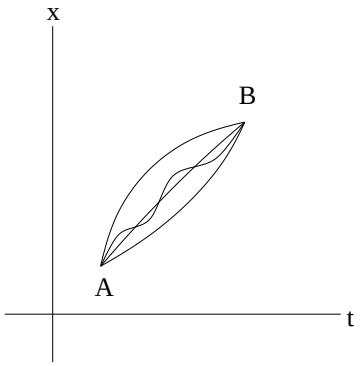


Figure 3.1: Optimal and non-optimal paths in spacetime

3.1.1 Prologue: Why $\int (K.E. - P.E.)dt = minimum$?

Example. In its simplest manifestations, Hamilton's principle is a mathematical expression of two fundamental physical aspects of nature: Newton's First Law of Motion and the Equivalence Principle. This can be seen from the following considerations:
A. Launch a particle vertically from x_1 at time t_1, watch it reach its maximum height, and then catch it at time t_2 at the instant it is located at x_2.

From Galileo we learned that in its travel from $(t_1,\, x_1)$ to $(t_2,\, x_2)$ the particle traces a space-time trajectory which is given by a parabola. Why so? Answer:

1. Newton's 1st Law: Every body continues in its state of rest, or of uniform motion in a straight line, unless it iscompelled to change that state by forces impressed upon it.

2. The principle of equivalence.

A. Simpler case: Free Particle.
Consider the motion of a particle moving freely in a free float ("inertial") frame. This particle moves with constant velocity, i.e. its space-time trajectory is a straight line.

The implication of this fact is that for such a curve the integral

$$\frac{1}{(t_2 - t_1)} \int_{t_i}^{t_2} \left(\frac{dx(t)}{dt} \right)^2 dt \equiv \langle v^2 \rangle = min\,!$$

as compared to other curves having the same starting and termination points.
Q: Why?

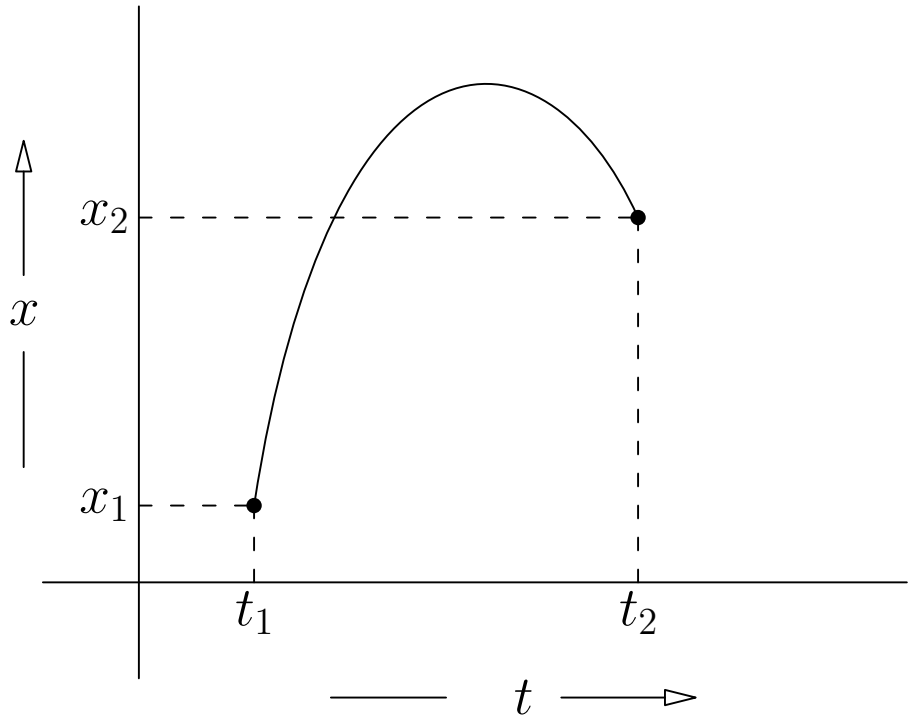


Figure 3.2: Spacetime trajectory of a particle thrown into the air.

A: All such curves have the same average velocity,

$$\langle \bar{v} \rangle = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \frac{d\bar{x}}{dt} dt = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} \frac{dx}{dt} dt = \langle v \rangle = v,$$

which means that the area under the curves $\bar{v}(t)$and $v(t) = v$ are the same.

Applying this fact to the positivity of the averaged squared deviation (away from the average),

$$0 \leqslant \langle (\bar{v} - \langle \bar{v} \rangle)^2 \rangle = \langle \bar{v}^2 \rangle - (\langle \bar{v} \rangle)^2 = \langle \bar{v}^2 \rangle - (\langle v \rangle)^2 ,$$

one has

$$\langle \bar{v}^2 \rangle \geqslant (\langle v \rangle)^2 = v^2,$$

or

$$\int_{t_i}^{t_2} \left(\frac{d\bar{x}(t)}{dt} \right)^2 dt \geqslant \int_{t_i}^{t_2} \left(\frac{dx(t)}{dt} \right)^2 dt.$$

This says that a free particle moves so that the integral of its kinetic energy is a minimum:

$$\int_{t_i}^{t_2} K.E.\, dt \equiv \int_{t_i}^{t_2} \frac{1}{2} m \left(\frac{dx(t)}{dt} \right)^2 dt = min\,!$$

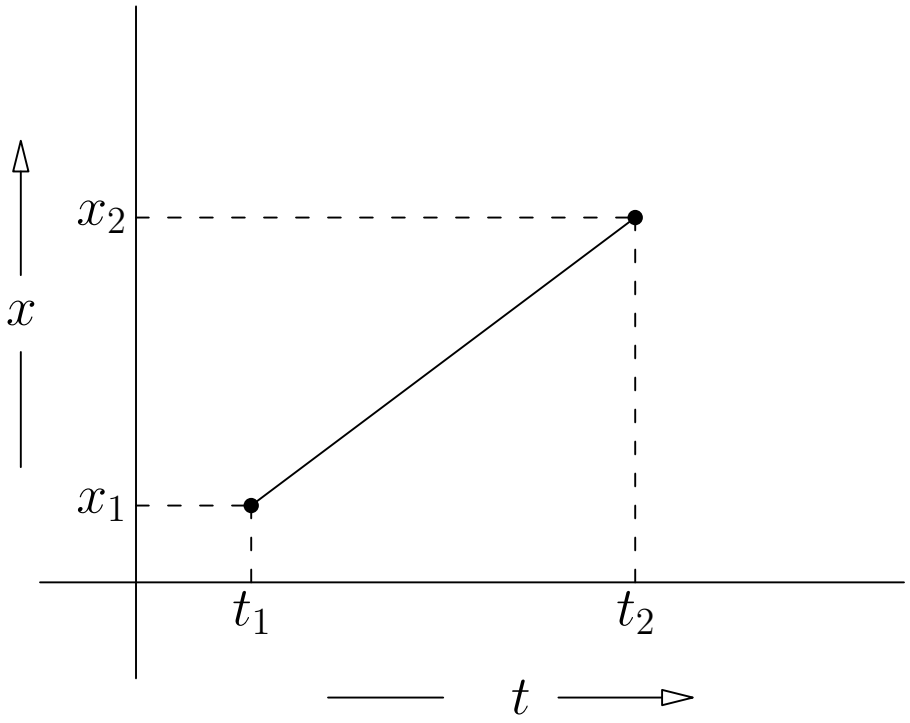
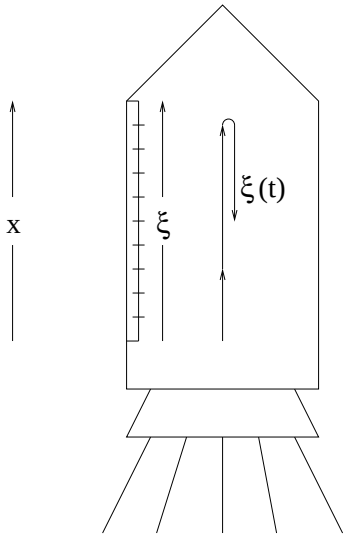


Figure 3.3: Spacetime trajectory of a free particle is a straight line.

B. Free particle in an accelerated frame.

Consider the motion of the same particle moving freely in a frame accelerated uniformly with acceleration g.



A point ξ fixed in the accelerated frame will move relative to the free float frame

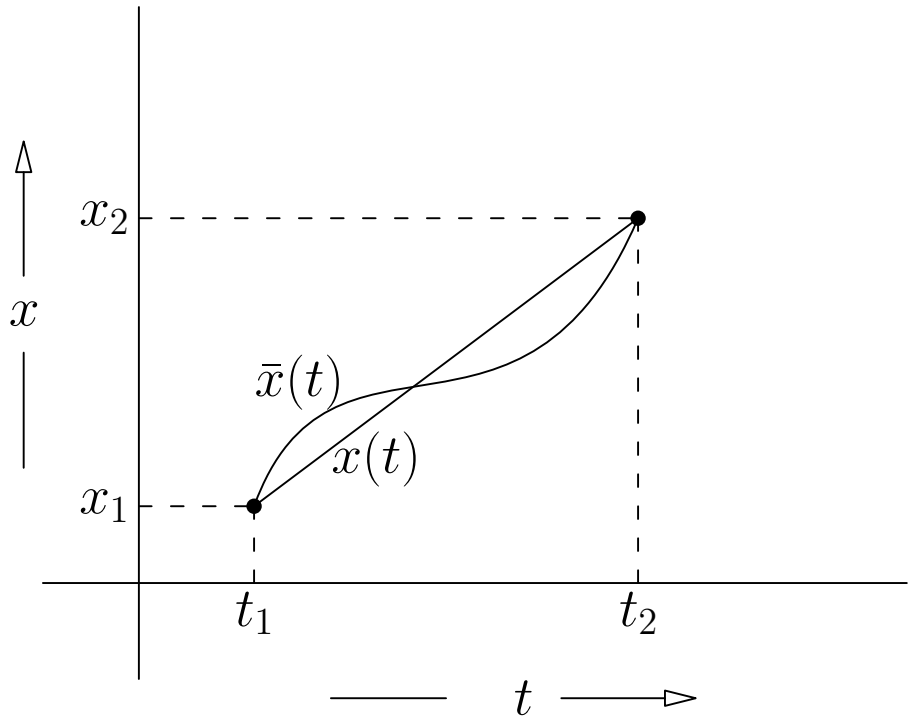


Figure 3.4: Straight line $x(t)$ and its variant $\bar{x}(t)$ have the same average velocity: $\langle \bar{v} \rangle = v \; (= const.)$.

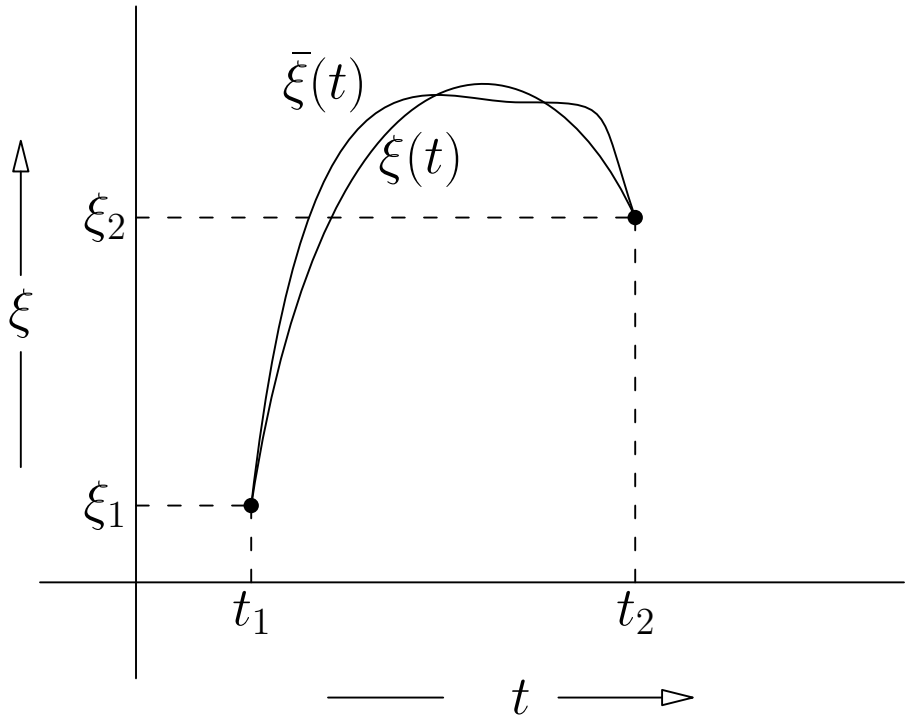


Figure 3.5: Minimizing trajectory $\xi(t)$ and one of its variants $\bar{\xi}(t)$.

according to

$$x = \xi + \frac{1}{2}gt^2.$$

It follows that, relative to the accelerated frame, the spacetime trajectory of the particle, $\xi(t)$, is given by

$$x(t) = \xi(t) + \frac{1}{2}gt^2. \tag{3.1}$$

Here $x(t)$ is the linear trajectory in Figure 3.3.

The to-be-minimized integral takes the form

$$\begin{aligned} min = \int_{t_i}^{t_2} \left(\frac{dx(t)}{dt}\right)^2 dt &= \int_{t_i}^{t_2} \left(\frac{d\xi}{dt} + gt\right)^2 dt \\ &= \int_{t_i}^{t_2} \left\{ \left(\frac{d\xi}{dt}\right)^2 + 2gt\frac{d\xi}{dt} + g^2t^2 \right\} dt \\ &= \int_{t_i}^{t_2} \left\{ \left(\frac{d\xi}{dt}\right)^2 - 2g\xi \right\} dt + 2gt\xi|_{t_1}^{t_2} + \frac{1}{3}gt^2|_{t_1}^{t_2} \end{aligned}$$

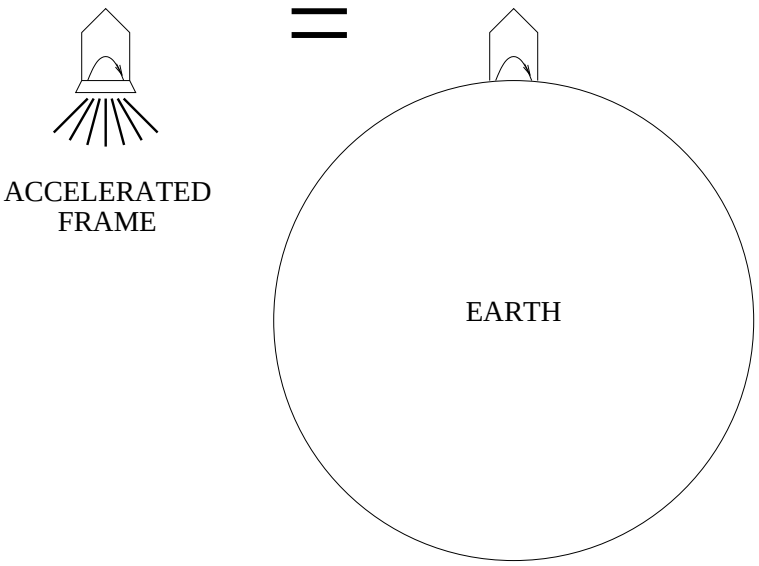


Figure 3.6: Trajectories in an accelerated frame are indistinguishable from those in a gravitational field. In particular the motion of particles of different composition (gold, aluminum, snakewood, etc.) is independent of their composition.

The last line is the result of an integration by parts. The last two terms are the same for all trajectories passing through the given points $(t_1,\, x_1)$ and $(t_2,\, x_2)$. Consequently,

$$\int_{t_i}^{t_2} \frac{1}{2} m \left(\frac{dx(t)}{dt} \right)^2 dt = min \iff \int_{t_i}^{t_2} \left\{ \frac{m}{2} \left(\frac{d\xi}{dt} \right)^2 - mg\xi \right\} dt = min$$

C. Free particle in an *equivalent* gravitational field.

The *equivalence principle* is an observation of the fact that in an accelerated frame the laws of moving bodies are the same as those in a homogeneous gravitational field.

Recall that in a gravitational field

$$mg\xi = P.E.$$

represents the potential energy of a mass m at a height ξ. Consequently, the trajectory of a particle in a gravitational field is determined by

$$\int_{t_i}^{t_2} (\, K.E. \, - P.E. \,)\, dt \equiv \int_{t_i}^{t_2} L(\dot{x},\, x,\, t)\, dt = min.$$

In fact, the trajectory of a particle which satisfies this minimum condition satisfies the Euler-Lagrange

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x},$$

which is Newton's second law of motion

$$ma = F$$

for the one-dimensional motion of a particle.
Nota bene:

1. The same minimum priciple hold even if g, and hence the potential energy $P.E.$, depends explicitly on time.

2. This principle is a special case of what is known as Hamilton's principle of least action. The difference is that the latter also accomodates motion which are subject to constraints.

D. Extension to multi dimensions and generic potentials.

The Lagrangian formulation opens new vistas on the notion of bodies. It can be fruitfully implemented for more general motions and potentials. These generalizations are alternate but equivalent formulations of Newtonian mechanics. They are simply expressed by the statement that

$$\int_{t_i}^{t_2} (\, K.E. \, - P.E. \,)\, dt = min$$

with

$$K.E. = \frac{1}{2} \sum_{i=1}^{n} m_i \dot{\vec{x}}_i \, \cdot \, \dot{\vec{x}}_i$$

$$P.E. = U(t, \, \vec{x}_i)$$

on the class of all system trajectories having fixed endpoints.

The advantage of Lagrangian Mechanics becomes evident in the process of setting up Newton's equations of motion. In Newtonian Mechanics one must do this for each force component separately, a task which becomes non-trivial relative to curvilinear coordinate frames (spherical, cylindrical, etc.). By contrast, in the Lagrangian approach one merely identifies the two scalars $K.E$ and $P.E.$ relative to the coordinate frame of one's choice. The remaining task of setting up the differential equations of motion is done automatically by merely writing down the Euler-Lagrange equations.

In terms of the calculus of variations, the principle of stationary action is simply

$$\int_{B}^{A} (T - U) dt = \text{extremum!}$$

Here $T(\dot{x}^i)$ is the kinetic energy (*K.E.*) and $U(x^i, t)$ is the potential energy (*P.E.*). The integrand $T(\dot{x}^i) - U(x^i, t) \equiv L(\dot{x}^i, x^i, t)$ is the *Lagrangian* of the system. Thus the stationary action principle can be stated as

$$\delta \int_A^B L(x^i, \dot{x}^i, t)dt = 0\,.$$

3.1.2 Hamilton's Principle: Its Conceptual Economy in Physics and Mathematics

Hamilton's principle lends itself to a treatment within the frame work of the calculus of variations if one identifies the action with the variational integral

$$J[y_i] = \int_{x_0}^{x_1} F(x, y_i, y_i')dx$$

By making the identificaiton

$$\begin{aligned} x &\rightarrow t \\ y_i(x) &\rightarrow x^i(t) \\ y_i'(x) &\rightarrow \dot{x}^i(t) \\ F(x, y_i, y_i^{'}) &\rightarrow L(x^i, \dot{x}^i, t) \end{aligned}$$

one obtains the Euler-Lagrange equations,

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0\,.$$

These equations of motion reproduce the Newtonian equations of motion. This fact is illustrated by means of the following example. For n particles, each with mass m_i one has relative to cartesian coordinates (x,y,z)

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right)$$

and

$$U = U(t, x_1, y_1, z_1, \cdots, x_n, y_n, z_n)\,.$$

The Euler-Lagrange equations are

$$\begin{aligned} -\frac{\partial U}{\partial x_i} &= m_i\ddot{x}_i \\ -\frac{\partial U}{\partial y_i} &= m_i\ddot{y}_i \\ -\frac{\partial U}{\partial z_i} &= m_i\ddot{z}_i \qquad i = 1, 2, 3 \end{aligned}$$

Thus one has $3n$ differential equations for $3n$ unknown functions. These equations are Newton's equations of motion for a system of n particles because

$$\left(-\frac{\partial U}{\partial \dot{x}_i}, -\frac{\partial U}{\partial \dot{y}_i}, -\frac{\partial U}{\partial \dot{z}_i}\right) = \overrightarrow{Force} \text{ on the ith particle}\,.$$

The advantage of the variational formulation of mechanics via Hamilton's principle over Newton's formulation is twofold:

1. From the viewpoint of physics it opens up new vistas. In particular, Hamilton's principle is a *bridge to quantum mechanics*. In fact, it is quantum mechanics which sanctions and justifies Hamilton's principle of least action. We shall return to that point later.

2. From the viewpoint of mathematics this formulation of mechanics constitutes an enormous streamlining, both in setting up and in solving the equations of motion for complex Newtonian dynamical systems.

   (a) Setting up the equations has been reduced to constructing a single scalar function, the Lagrangian, and its derivatives.

   (b) Solving the resulting system of differential equations is greatly facilitated by the flexibility inherent in a scalar function. The function, and hence the system of equations, can be exhibited relative to any system of coordinates one chooses, including those relative to which the equations are so simple that they can be readily analysed, if not solved.

The conceptual unit economy, and hence the technical advantage and the mathematical power of Hamilton's principle, arises from the fact that the Lagrangian is a single scalar

$$L = K.E. - P.E.\,,$$

and from the fact that it may be expressed in terms of virtually any type of coordinates. These "generalized coordinates"

$$q^1(t), q^2(t), \cdots, q^s(t)$$

and their time derivatives

$$\dot{q}^1(t), \dot{q}^2(t), \cdots, \dot{q}^s(t)$$

characterize the state of motion of the mechanical system. Quite often they are (but need not be) related to, say, the cartesian coordinates, $\vec{x} = (x, y, z)$ by

$$\vec{x}_i = \vec{x}_i(q^1, \cdots, q^s, t) = \vec{x}_i(q^j, t)$$

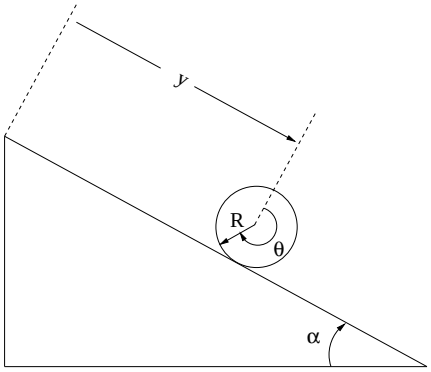


Figure 3.7: Disk rolling down an inclined plane

and their time derivative

$$\dot{\vec{x}}_i = \dot{\vec{x}}_i(\dot{q}^j, q^k, t)\,.$$

In any case, in terms of generalized coordinates Hamilton's principle becomes

$$\delta \int_{t1}^{t_2} L(q^j, \dot{q}^j, t) = 0\,.$$

The associated Euler-Lagrange equations of motion are

$$\frac{\partial L}{\partial q^j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^j}\,. \tag{3.2}$$

The ease with which the equations of motion can be set up is illustrated by the following simple

Example. Consider a disk rolling down a plane of lenth ℓ inclined at an angle α relative to the horizontal . The task of setting up the equations for the system is archetypical:

1. The total kinetic energy breaks up into its translational and its rotational part (around the center of mass)

$$\begin{aligned} T &= \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 \\ &= \frac{1}{2}M\dot{y}^2 + \frac{1}{4}MR^2\dot{\theta}^2 \end{aligned}$$

   Here $I = \frac{1}{2}MR^2$ is the moment of inertia of the disc whose radius is R and whose mass is M.

2. The potential energy is

$$U = Mg\,(\ell - y)\,\sin(\alpha)$$

so that U=0 at the bottom ($y = \ell$) of the inclined plane.

3. The Lagrangian is

$$L = T - U = \frac{1}{2}M\dot{y}^2 + \frac{1}{4}MR^2\dot{\theta}^2 + Mg(y-\ell)\sin(\alpha)\,.$$

4. The equation of constraint between tranlation and rotation is

$$G(y,\theta) = y - R\theta = 0\,.$$

5. Finally, the Euler-Lagrange equations are

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} &= \lambda(t)\frac{\partial G}{\partial y}\\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= \lambda(t)\frac{\partial G}{\partial \theta}\end{aligned}\quad .$$

Explicitly, they are

$$M\ddot{y} - Mg\,\sin\alpha = \lambda \tag{3.3}$$

$$\frac{1}{2}MR^2\ddot{\theta} = -\lambda R\,. \tag{3.4}$$

The constraint is

$$y = R\theta\,. \tag{3.5}$$

By differentiating the last equation, using the result to rewrite the antecedent equation, and introducing the resultant expression for λ, these three equations yield

$$\begin{aligned}\ddot{\theta} &= \frac{\ddot{y}}{R}\\ \lambda &= -\frac{1}{2}M\ddot{y}\\ \ddot{y} &= \frac{2g\,\sin\alpha}{3}\,.\end{aligned}$$

Consequently,

$$\begin{aligned}\lambda &= \frac{-Mg\,\sin\alpha}{3}\\ \ddot{\theta} &= \frac{2g\,\sin\alpha}{3R}\,.\end{aligned}$$

Remark. Note that the introduction of the Lagrange multiplier λ into the variational principle reveals physical aspects of the mechanical system which would have stayed hidden without that multiplier. In this example we see that on the right hand side of Eq.(3.3), λ is a *constraint force*, and on the right hand side of Eq.(3.4), $-\lambda R$ is a constraint torque. These constraint-induced quantities (i.e. λ and -λR) are non-zero in order to guarantee that the constraint, namely, *no slippage*, stays enforced. In particular, we see that λ is a constraint force along the y-direction. Furthermore, we see that -λR in Eq.(10.8b) is a positive torque. This guarantees that the disk spins up in the positive θ direction.

3.2 Hamilton-Jacobi Theory

Lecture 10

Newton created mechanics and formulated it in terms of his differential laws of motion. Using Hamilton's Principle, Lagrange streamlined Newton's formulations by giving it an innovative mathematical formulation which is based on a single scalar, the Lagrangian. As a consequence, there is an infinite flexibility in describing the mechanical system; in other words, the equations of motion can be exhibitied relative to any chosen set of generalized coordinates.

Hamilton-Jacobi theory, which also springs forth from Hamilton's principle, constitutes an additional step forward in physics as well as in mathematics. The advance in mathematics consists of the fact that H-J theory constitutes a powerful and systematic way of integrating the equations of motion of mechanics. The advance in physics consists of the fact that H-J theory provides a bridge between classical mechanics and quantum mechanics, as well as between ray optics and electromagnetic wave theory (wave optics).

It is difficult to point to a mathematical theory which brings into our grasp a larger and more diverse number of aspects of nature than H-J theory. No wonder many physicists express their admiration and enthusaism for H-J theory in a way which borders on the poetic. For example, Cornelius Lanczos in his book "The Variational Principles of Mechanics" (1949), writes: "We now are in the rarified atmosphere of theories of excessive beauty and we are nearing a high plateau on which geometry, optics, mechanics, and wave mechanics meet on common ground. Only concentrated thinking, and a considerable amount of re-creation, will reveal the full beauty of our subject in which the last words have not yet been spoken. We start with the integration theory of Jacobi and continue with Hamilton's own investigations in the realm of goemetrical optics and mechanics. The combination of these two approaches leads to de Broglie's and Schroedinger's great discoveries, and we come to the end our journey."

3.3 The Dynamical Phase

The equations of motion of a dynamical system are obtained from Hamilton's principle, namely the require ment that the action functional

$$I = \int_{x',t'}^{x,t} L\left(x, \frac{dx}{dt}, t\right) dt \tag{3.6}$$

be an extremum for the actual path along which the system travels in space and in time. This actual path we call an *extremal* path. Suppose we consider only extremal paths, in particular those that have the same common starting point (x', t') in space and time. The termination point of each extemal path is different, and we designate it by the variable point (x, t). For of these extremal path the action integral has a (unique) extemal value, namely

$$\int_{x',t'}^{x,t} L\left(x, \frac{dx}{dt}, t\right) dt = I_{extremum}\,.$$

For each endpoint (x, t) of an extemal path there is number $I_{extremum}(x, t)$. Thus we have a real valued function which maps (x, t) into the reals:

$$S(x, t) \equiv I_{extemum}(x, t)\,.$$

The level surfaces (isograms) of this scalar function are easy to constuct. Suppose we wish to constuct the isogram

$$S(x, t) = 17.1\,.$$

We consider the set of extemal paths emanating from the common point (x', t'). At that point $S(x', t') = 0$. We now move along a particular extemal path and evaluate the action integral, Eq.(3.6), along this path. When the value of this integral is $I = 17.1$, we stop. This stopping point is a point on the isogram $S(x, t) = 17.1$.

The function defined by this construction is called the "dynamical phase", or the "action" of the system at (x, t). It also is called the "eikonal" in optics. This function is easy to remember because in wave mechanics it is the phase of the system. In fact, we assert that its relation to the wave function of the system is give by

$$\Psi(x, t) = \mathcal{A}e^{iS(x,t)/\hbar}.$$

Here $\hbar$ is a constant which guarantees that the exponent is dimensionless. For a mechanical system this constant is Planck's constant (6.1×10^{-27}erg sec) divided by 2π. The wave function Ψ satisfies a wave equation. Whether this wave equation is the Schroedinger equation of the system, the scalar wave equation for a light ray, or some other linear wave equation is of great importance, but of no concern to us at present. For us, our present interest lies only in the dynamical phase $S(x, t)$,

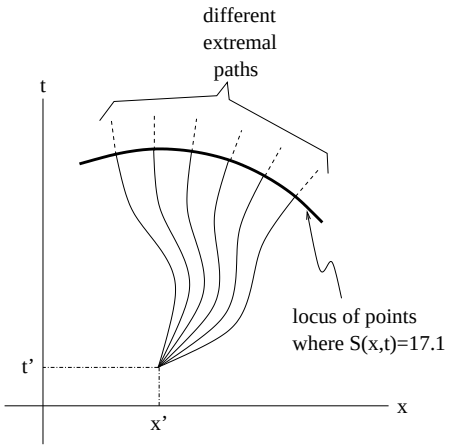


Figure 3.8: Isogram of the dynamial phase S(x,t)

1. in the equation for $S(x,t)$ (Section 3.5) and

2. in the reconstruction of the extremal paths (Sections 3.6 and 3.11) from $S(x,t)$.

The physical perspective on our reconstruction of the extremal paths is very different from that of Newton. Whereas his focus is on the particle aspects of bodies, ours is on their wave aspects. In spite of that difference our wave perspective started with Newton. It grew by a well-defined process of inductive reasoning consisting of Newton →Euler-Lagrange →Hamilton →Jacobi + experimental observations about the dynamical behavior of matter – observations which were performed during the first half of the twentieth century. In spite of the great conceptual distance between the particle and the wave perspectives, the former is a correspondence (asymptotic, shortwave length) limit of the latter. This fact is developed in Sections 3.6-3.11 and summarized in Figure 3.13 and 3.14 on pages 118 and 122 respectively.

3.4 Momentum and the Hamiltonian

We now proceed with the process of deriving the partial differential equation which the scalar function $S(x,t)$ satisfies. In this process we shall repeatedly encounter two quantities. They are (i) the *generalized momentum* and (ii) the *Hamiltonian* of the system. Specifically we let

$$\frac{\partial L}{\partial \dot{q}^i} \equiv p_i \qquad i = 1, ...s \tag{3.7}$$

be the ith generalized momentum. Its importance follows from the equation of motion. If the Lagrangian is independent of the ith generalized coordinate q^i, i.e.

$$\frac{\partial L}{\partial q^i} = 0$$

then the Euler-Lagrange equations of motion demand that

$$\frac{dp_i}{dt} = 0 \, .$$

In other words, the ith generalized momentum of the system does not change; this *momentum is "conserved".*

Example. Consider a system of n particles in a potential. The Lagrangian is

$$L = \frac{1}{2} \sum_{j=1}^{n} m_j \left(\dot{x}_j^2 + \dot{y}_j^2 + \dot{z}_j^2 \right) - U \left(x_j, y_j, z_j, t \right) \, .$$

The x-component of the jth particle *momentum* is

$$\frac{\partial L}{\partial \dot{x}_j} = m \dot{x}_j \equiv p_{xj} \, .$$

The other repeatedly occuring quantity is

$$\sum_{i=1}^{s} \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L \equiv H \, , \tag{3.8}$$

the *Hamiltonian* of the system. This quantity is conserved if the Lagrangian of the system has no explicit time dependence. Indeed, differentiating each term of that expression for the Hamiltonian and keeping in mind that L depends on q^i, $\dot{q}^i$, and t, one obtains

$$\frac{dH}{dt} = \sum_{i=1}^{s} \left[\ddot{q}^i \frac{\partial L}{\partial \dot{q}^i} + \dot{q}^i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} \dot{q}^i - \frac{\partial L}{\partial \dot{q}^i} \ddot{q}^i \right] - \frac{\partial L}{\partial t} \, .$$

Here the two middle terms cancel because of the E-L equations of motion. Consequently, we are left with

$$\frac{dH}{dt} = - \frac{\partial L}{\partial t} \, . \tag{3.9}$$

We conclude that if $\frac{\partial L}{\partial t} = 0$, then

$$H = constant$$

along the extremal path of the system.

Remark. If the system has only a single degree of freedom ($s = 1$) then $H = \dot{q}\frac{\partial L}{\partial \dot{q}} - L$, and Eq.(3.9) is simply the "2nd form of the Euler equation", which was considered at the end of Chapter I.

The generalized momenta p_j , $j = 1, \cdots, s$ as well as the Hamiltonian H will play the key roles in establishing the partial diffential equation which the dynamical phase $S(q^1, q^2, \cdots, q^s, t)$ must satisfy. In fact, in order to obtain this equation, we shall find that one must reexpress the Hamiltonian, Eq.(3.8), in terms of the generalized momenta defined by Eq.(3.7) in such a way that the Hamiltonian has no functional dependence on the generalized velocities $\dot{q}^j$, *j= 1,*$\cdots$*,s.* This goal is achieved by solving the defining equations (3.7) for the generalized velocities

$$\dot{q}^i = \dot{q}^i(q^k, p_j)$$

in terms of the momenta, and then introducing these expressions into the Hamiltonian:

$$\sum_{i=1}^{s} \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L = H(q^k, p_j, t)\,. \tag{3.10}$$

To verify that H does indeed depend only on q^k and p_j, but not on $\dot{q}^i$, we observe that[1]

$$\begin{aligned} dH &= d\left(\dot{q}^i p_i - L\right) \\ &= d\dot{q}^i p_i + \dot{q}^i dp_i - \frac{\partial L}{\partial q^i} dq^i - \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i - \frac{\partial L}{\partial t} dt\,. \end{aligned}$$

The first and the fourth term cancel with the help of Eq.(3.7). What remains is

$$dH = \dot{q}^i dp_i - \frac{\partial L}{\partial q^i} dq^i - \frac{\partial L}{\partial t} dt\,.$$

There are two benefits. First, this shows that, with the help of the defining relation (3.7), the Hamiltonian is functionally independent of $\dot{q}^j$, and depends only on q^k and p_j instead. Second, by comparing this expression with the differential of Eq.(3.10),

$$dH = \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial t} dt\,,$$

by using the E-L equations of motion (3.2),

$$\frac{\partial L}{\partial q^i} = \frac{d}{dt}\, p_i\,,$$

[1]In the interest of conceptual unit economy we are introducing the Einstein summation convention according to which a summation is implied whenever a pair of repeated (summation!) indices occur in a product expression. For example, $\sum_{i=1}^{s} \dot{q}^i p_i \equiv \dot{q}^i p_i$.

and by taking advantage of the linear independence of the differentials $\{dp_i, dq^i, dt\}$, one obtains the three equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \quad \text{and } \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}\,.$$

These are *Hamilton's equations of motion.* They govern the dynamics of the mechanical system. Its motion is represented by the trajectory of a moving point $\{q^i(t), p_i(t)\}_1^s$ in the $2s$-dimensional space spanned by $\{q^i, p_i\}$. The rate of change of the Hamiltonian function $H(q^i, p_i, t)$ along the trajectory is

$$\begin{aligned}\frac{dH}{dt} &= \frac{\partial H}{\partial q^i}\dot{q}^i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t} \\ &= \frac{\partial H}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q^i} + \frac{\partial H}{\partial t} \\ &= \frac{\partial H}{\partial t}\,.\end{aligned}$$

Thus, if t is a cyclic coordinate, i.e. the Hamiltonian is manifestly independent of time ($\partial H/\partial t = 0$), then H is evidently a constant of motion.

The Hamiltonian equations of motion are equivalent to those of Euler-Lagrange. The important difference is that the former is a system of *2s first* order equations, while the latter is one of *s second* order equations.

Example. (*Particles in the Field of a Potential)*

The Lagrangian is

$$L = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{\vec{x}}_i \dot{\vec{x}}_i - U(\vec{x}_k, t)\,.$$

For each of the particles the momentum is

$$\vec{p}_i = (p_{xi},\, p_{yi},\, p_{zi}) = m_i \dot{\vec{x}}_i\,. \tag{3.11}$$

The Hamiltonian is

$$\begin{aligned}H &= \sum_j \dot{q}^j \frac{\partial L}{\partial q_j} - L \\ &= \sum_{i=1}^{n} (\dot{x}_i m_i \dot{x}_i + \dot{y}_i m_i \dot{y}_i + \dot{z}_i m_i \dot{z}_i) - \frac{1}{2}\sum_{i=1}^{n} (\dot{x}_i m_i \dot{x}_i + \dot{y}_i m_i \dot{y}_i + \dot{z}_i m_i \dot{z}_i) + U(\vec{x}_k, t) \\ &= \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\vec{x}}_i \cdot \dot{\vec{x}}_i + U(\vec{x}_k, t) \\ &= K.E. + P.E. = \text{Total Energy}\,.\end{aligned}$$

With the momentum defined by Eq. (3.11), the Hamilitonian assumes the form

$$H(\vec{p}_i, \vec{x}_i, t) = \sum_{i=1}^{n} \frac{\vec{p}_i \cdot \vec{p}_i}{2m_i} + U(\vec{x}_k, t)\,,$$

and the two sets of Hamiltonian equations of motion are

$$\begin{aligned} \frac{d\vec{x}_i}{dt} &= \frac{\vec{p}_i}{m} \\ \frac{d\vec{p}_i}{dt} &= -\vec{\nabla}_i\, U(\vec{x}_k, t)\,. \end{aligned}$$

If the potential energy is time-independent and hence depends only on the position $\vec{x}_i$ of the particles, then H is cyclic in t:

$$0 = \frac{\partial H}{\partial t}\,.$$

Consequently,

$$H = T.E. = \text{constant}\,.$$

Thus the total energy of a system of particles moving in a time independent potential is a constant of motion, a conclusion one also infer directly from the fact that the Lagrangian does not contain any explicit dependence on time t.

Remark. That the Hamiltonian is the total energy of the system is true in a non-rotating coordinate frame. However, in a rotating coordinate frame the Hamiltonian is not equal to the system's total energy. However, if the Lagrangian has no explicit time dependence ("time invariant system") then the Hamiltonian is conserved nevertheless. To illustrate the issue, consider the above example of particles-in-a-potential problem from the perspective of a rotating frame, relative to which one achieves substantial simplifcation in the analysis of charged particles circulating in a magnetic field or of satellites in an earth-centered reference frame.

Problem. (*Particle motion in a rotating frame)*
Given a frame (x', y', z') which rotates with constant angular frequency ω relative to a given inertial frame (*x,y,z*),

$$\begin{aligned} x &= x' \cos\omega t - y' \sin\omega t \\ y &= x' \sin\omega t + y' \sin\omega t \\ z &= z' \end{aligned}$$

determine for a single particle in a rotationally symmetric potential $U\,(x^2 + y^2,\, z)$ (i) its Lagrangian, (ii) its Hamiltonian relative the rotating frame, and then (iii)

compare the two Hamiltonians in the two frames.
Solution. The Lagrangian of a particle moving in the potential U is

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U\left(x^2 + y^2,\, z\right) .$$

Relative to the rotating frame this Lagrangian,

$$L = \frac{m}{2}\left(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2\right) - U\left(x'^2 + y'^2,\, z'\right) + \frac{m}{2}\omega^2\left(x'^2 + y'^2\right) + \omega m\left(x'\dot{y}' - y'\dot{x}'\right),$$

has acquired an additional centrifugal potential energy, and a velocity dependent potential energy. In the rotating frame the Hamiltonian is

$$\begin{aligned} H' &= \dot{x}'\frac{\partial L}{\partial \dot{x}'} + \dot{y}'\frac{\partial L}{\partial \dot{y}'} + \dot{z}'\frac{\partial L}{\partial \dot{z}'} - L \\ &= \frac{m}{2}\left(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2\right) + U\left(x'^2 + y'^2,\, z'\right) - \frac{m}{2}\omega^2\left(x'^2 + y'^2\right) . \end{aligned}$$

Introducing the defining relations, Eq.(3.7)

$$\begin{aligned} p'_x &= m\dot{x}' - m\omega y' \quad &\rightarrow \dot{x}' = \frac{p'_x}{m} + \omega y' \\ p'_y &= m\dot{y}' + m\omega x' \quad &\rightarrow \dot{y}' = \frac{p'_y}{m} - \omega x' \\ p'_z &= m\dot{z}' \quad &\rightarrow \dot{z}' = \frac{p'_z}{m} , \end{aligned}$$

one finds that this Hamiltonian is

$$H' = \frac{(p'_x)^2}{2m} + \frac{(p'_y)^2}{2m} + \frac{(p'_z)^2}{2m} + U\left(x'^2 + y'^2,\, z'\right) - \omega\left(x'p'_y - y'p'_x\right).$$

The particle's inertial Hamiltonian

$$\begin{aligned} H &= \frac{(p_x)^2}{2m} + \frac{(p_y)^2}{2m} + \frac{(p_z)^2}{2m} + U\left(x^2 + y^2,\, z\right) \\ &= \text{Kinetic Energy} + \text{Potential Energy} \end{aligned}$$

expresses its total energy. It is an integral of motion. By contrast the particle's rotational Hamiltonian,

$$H' = (\text{Kinetic Energy})' + (\text{Potential Energy})' - \omega L_z ,$$

even though it is also an integral of motion, does not express a mere total energy, kinetic plus potential in the rotating frame. There is an additional energy-like quantity

$-\omega L_z$ which expresses a kind of interaction between (a) whatever aspect (proportional to ω) is responsible for the curved meniscus in Newton's bucket experiment[2] and (b) the particle's moment of momentum (i.e. angular momentum),

$$\begin{aligned} L_z &= \left(x' p'_y - y' p'_x\right) \\ &= \left(x\, p_y - y\, p_x\right) \end{aligned}$$

It is evident that this quantity is the same in the rotating and the inertial frame. It therefore refers to a property independent of any rotational aspect of the frame. Furthermore, it is an integral of motion, a fact which is implied by Hamilton's equations of motion. Thus there are three integrals of motion: (i) H', (ii) (K.E.)$'$+(P.E.)$'$, (iii) L_z and any two imply the third.

3.5 The Hamilton-Jacobi Equation

We now return to the task of determining the equation that governs the dynamical phase for a system with a single degree of freedom. The extension to several degrees takes little effort by comparison.

3.5.1 Single Degree of Freedom

Consider a set of extremal paths emanating from event A, i.e. they all start at the same time t' at the same point x' in space. These paths being extremal, they all satisfy the same Euler-Lagrange equation. They also pass through the same starting event, but they have different initial velocities. We evaluate the "action" along each extremal path,

$$I_{extremum}(t, t') = \int_{t'}^{t} L\left(x(\lambda), \dot{x}(\lambda), \lambda\right) d\lambda \,.$$

Consider the locus of points where, on different extremals, the integral $I_{extreme}$ has the same value, say

$$I_{extremum} = I_{extremum.}(t, t') \,.$$

This locus of points forms the level surfaces of a function *S(x,t)* where

$$S(x,t) = \text{extremum value of } \int_{x',t'}^{x,t} L(x, \dot{x}, t) dt \,.$$

[2]Newton considered a water-filled bucket rotating (around its axis) relative to a free float (i.e. inertial) frame – a frame relative to which all free bodies obey Newton's first law of motion, but a frame which today we recognize as having the additional feature of not rotating relative to the fixed stars. The rotating frame associated with this bucket is characterized by a single distinguishing feature, its angular velocity ω. In this frame the surface of unmoving water curves upward with increasing radius. This curvature is absent in the inertial frame, which is not rotating relative to the fixed stars.

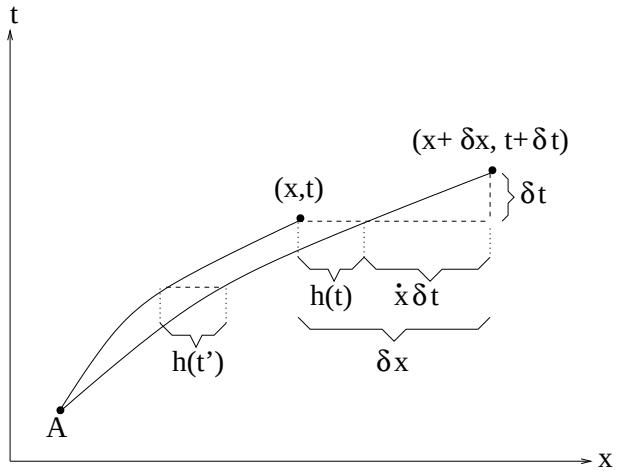


Figure 3.9: Two nearby extremal paths.

We shall now see that the partial derivatives of this scalar function are the momenta and the negative Hamiltonian (usually, but not always, the energy), i.e.

$$\delta S = p\delta x - H\delta t$$

We consider two *extemal* paths. They have the same starting point A. But their end points are respectively (x,t) and (x+δx,t+δt). The difference in the dynamical phase (action) of these endpoints is

$$\begin{aligned}
\delta S &= \delta I_{extremum} \\
&= \int_A^{t+\delta t} L(x+h, \dot{x}+\dot{h}, t)dt - \int_A^t L(x,\dot{x},t)dt \\
&= L\delta t + \int_A^t \left(\frac{\partial L}{\partial x}h + \frac{\partial L}{\partial \dot{x}}\dot{h}\right) dt \\
&= L\delta t + \frac{\partial L}{\partial \dot{x}}h + \int_A^t \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right) h\, dt\,.
\end{aligned}$$

We only consider extremal curves. Thus $\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$, and the integral vanishes. The variations δx and δt refer to the coordinate differences of arbitrarily located endpoint of the two extremals. But the variation $h(t)$ connects pairs of points having the same curve parameter t. Consequently, as one can see in Fig. 3.9,

$$h(t) = \delta x - \dot{x}\delta t\,.$$

Thus

$$\begin{aligned}
\delta S &= \frac{\partial L}{\partial \dot{x}}\delta x - \left(\frac{\partial L}{\partial \dot{x}}\dot{x} - L\right)\delta t \\
&\equiv p\delta x - H\delta t
\end{aligned}$$

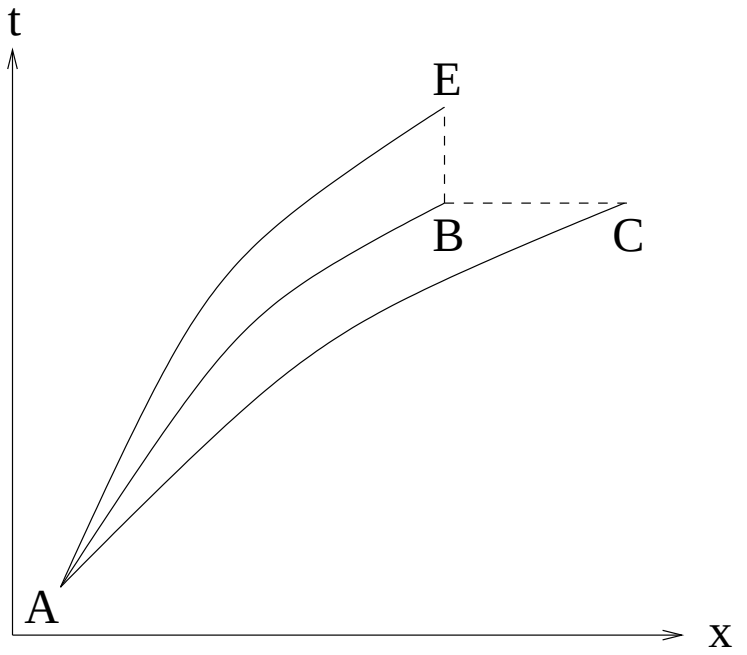


Figure 3.10: Three extremal curves

An inquiry as to how S(x,t) changes into the x-direction, keeping t fixed, yields

$$\left(\begin{array}{c}\text{rate of change of}\\ \text{dynamical phase}\\ \text{with position}\end{array}\right) = (\text{momentum}) \tag{3.12}$$

$$= \lim_{\substack{C\to B\\ t=fixed}} \frac{\delta S}{\delta x} = \frac{\partial S}{\partial x} \tag{3.13}$$

$$= \frac{\partial L(x,\dot{x},t)}{\partial \dot{x}} = p\,. \tag{3.14}$$

Similarly,

$$-\left(\begin{array}{c}\text{rate of change of}\\ \text{dynamical phase}\\ \text{with time}\end{array}\right) = (\text{energy}) \tag{3.15}$$

$$= -\lim_{\substack{E\to B\\ x=fixed}} \frac{\delta S}{\delta t} = -\frac{\partial S}{\partial t} \tag{3.16}$$

$$= \frac{\partial L}{\partial \dot{x}}\dot{x} - L = H\,. \tag{3.17}$$

These two equations, yield the "dispersion relation" at (x,t). Indeed, in Eq. (3.17) replace the $\dot{x}$-dependence with a dependence on p by solving Eq. (3.14) for $\dot{x}$. The

result is

$$(\text{energy}) = H(x, p, t)\,.$$

Finally introduce into this equation the partial derivatives calculated in Eqs. (3.13) and (3.16):

$$-\frac{\partial S}{\partial t} = H\left(x, \frac{\partial S}{\partial x}, t\right)\,.$$

This is the Hamilton-Jacobi equation, a single first order differential equation for the dynamical phase $S(x,t)$. This equation is a dispersion relation because it expresses how the time rate of change of the phase depends on its rate of change with position. For a system with several degrees of freedom the H-J equation is

$$H\left(q^i, \frac{\partial S}{\partial q^j}, t\right) + \frac{\partial S}{\partial t} = 0.$$

Lecture 11

Problem. (*Particle in a potential*)

Set up and solve the Hamilton-Jacoby equation for a particle in a one dimensional potential $U(x)$.

Solution. Setting up the H-J equation is a three step process.

1. Exhibit the Lagrangian:
$$L = \frac{1}{2}m\dot{x}^2 - U(x)\,.$$

2. Determine the momentum and the Hamiltonian:
$$\begin{aligned} p &= \frac{\partial L}{\partial \dot{x}} \\ &= m\dot{x}\,; \\ H &= \dot{x}\frac{\partial L}{\partial \dot{x}} - L \\ &= \frac{1}{2}m\dot{x}^2 + U(x)\,. \end{aligned}$$

3. Express the Hamiltonian in terms of the momentum:
$$H = \frac{p^2}{2m} + U(x)\,.$$

4. Write down the H-J equation $-\frac{\partial S}{\partial t} = H\left(x, \frac{\partial S}{\partial x}\right)$:
$$-\frac{\partial S}{\partial t} = \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + U(x)\,.$$

This a first order non-linear partial differential equation that needs to be solved for the scalar function $S(x,t)$.
This p.d.e. lends itself to being solved by the method of separation of variables according to which one finds solutions of the form

$$S(x,t) = T(t) + X(x)\,. \tag{3.18}$$

Introducing this form into the H-J equation, one finds

$$-\frac{dT(t)}{dt} = \frac{1}{2m}\left(\frac{dX(x)}{dx}\right)^2 + U(x)\,.$$

This equation says that the left hand side is independent of x, while the right hand side is independent of t. Being equal, the l.h.s. is also independent of x. Being independent of both t and x, it is a constant. Letting this "separation" constant be equal to E, one obtains two equations

$$\begin{aligned} -\frac{dT(t)}{dt} &= E \\ \frac{1}{2m}\left(\frac{dX(x)}{dx}\right)^2 + U(x) &= E\,. \end{aligned}$$

These are two ordinary equations for T and X. Inserting these equations into Eq. (3.18), one obtains the sought after solution to the H-J equation,

$$S(x,t) = -Et + \int^x \sqrt{2m\,(E - U(x'))}dx' + \delta(E)\,.$$

Here the "integration constant" $\delta(E)$ is an arbitrary function of E. Furthermore, observe that S depends on E also. This means that one has an E-parametrized family of solutions. Thus, properly speaking, separation of variables yields many solutions to the H-J equation, in fact, a one-parameter family of them

$$S(x,t) = S_E(x,t)\,.$$

3.5.2 Several Degrees of Freedom

We shall see in a subsequent section that whenever the H-J for a system with several degrees of freedom, say $\{q^i\}$, lends itself to being solved by the method of the separation of variables, i.e.

$$S(q^i,t) = T(t) + \sum_{i=1}^{s} Q_i(q^i)\,,$$

the solution has the form

$$S = -\int^t E dt + \sum_{i=1}^{s} \int^{q^i} p_i(x^i; E, \alpha_1, \cdots, \alpha_{s-1}) dq^i + \delta\left(E, \alpha_1, \cdots, \alpha_{s-1}\right)$$

Here δ is an arbitrary function of E and the other separation constants that arise in the process of solving the H-J equation. We see that for each choice of $(E, \alpha_1, \cdots, \alpha_{s-1})$ we have a different solution S. Thus, properly speaking, we have $S_{E,\alpha_1,\cdots,\alpha_{s-1}}$, a multi-parametrized family of solutions to the H-J equation.

We shall now continue our development and show that Hamilton-Jacobi Theory is
a) A new and rapid way of integrating the E-L equations
b) The bridge to wave (also "quantum") mechanics.
The virtue of Hamilton's principle is that once the kinetic and potential energy of the system are known, the equations of motion can be set up with little effort. These Euler-Lagrange equations are Newton's equations of motion for the system. Although setting up the equations of motion for a system is a routine process, solving them can be a considerable challenge. This task can be facilitated considerably by using an entirely different approach. Instead of setting up and solving the set of coupled Newtonian ordinary differential equations, one sets up and solves a single partial differential equation for a single scalar function. Once one has this scalar function, one knows everything there is to know about the dynamical system. In particular, we shall see that by differentiating this scalar function (the dynamical phase, the Hamilton-Jacobi function, the eikonal) one readily deduces all possible dynamical evolutions of the system.

3.6 Hamilton-Jacobi Description of Motion

Hamilton-Jacobi theory is an example of the *principle of unit economy*[3], according to which one condenses a vast amount of knowledge into a smaller and smaller number of principles. Indeed, H-J theory condenses all of classical mechanics and all of wave mechanics (in the asymptotic high-frequency/short-wavelength (a.k.a. W.K.B.) approximation) into two conceptual units,, (i) the H-J equation and (ii) the

[3]The *principle of unit economy*, also known informally as the "crow epistemology", is the principle that stipulates the formation of a new concept

1. when the description of a set of elements of knowledge becomes too complex,
2. when the elements comprising the knowledge are used repeatedly, and
3. when the elements of that set require further study.

Pushing back the frontier of knowledge and successful navigation of the world demands the formation of a new concept under any one of these three circumstances.

principle of constructive interference. These two units are a mathematical expression of the fact that classical mechanics is an asymptotic limit of wave mechanics.

Hamilton thinking started with his observations of numerous known analogies between "particle world lines" of mechanics and "light rays" of geometric optics. These observations were the driving force of his theory. With it he developed classical mechanics as an asymptotic limit in the same way that ray optics is the asymptotic limit of wave optics. Ray optics is a mathematically precise asymptotic limit of wave optics. Hamilton applied this mathematical formulation to classical mechanics. He obtained what nowadays is called the Hamilton-Jacobi formulation of mechanics. Even though H-J theory is a mathematical limit of wave mechanics, in Hamilton's time there was no logical justification for attributing any wave properties to material particles. (That justification did not come until experimental evidence to that effect was received in the beginning of the 20th century.) The most he was able to claim was that H-J theory is a mathematical method with more unit economy than any other formulation of mechanics. The justification for associating a wave function with a mechanical system did not come until observational evidence to that effect was received in the beginning of the 20th century.

We shall take advantage of this observation (in particular by Davidson and Germer, 1925) implied association by assigning to a mechanical system a wave function. For our development of the H-J theory it is irrelevant whether it satisfies the Schroedinger, the Klein-Gordon, or some other quantum mechanical wave equation. Furthermore, whatever the form of the wave equation governing this wave function, our focus is only on those circumstances where the wave function has the form

$$\Psi_E(x,t) = \underbrace{\mathcal{A}(x,t)}_{\textit{slowly}\text{ varying function of } x \text{ and } t} \times \underbrace{\exp\left(\frac{i}{\hbar}S_E(x,t)\right)}_{\textit{rapidly}\text{ varying function of } x \text{ and } t} \tag{3.19}$$

This circumstance is called the "high frequency" limit or the "semi-classical" approximation. It can be achieved by making the energy E of the system large enough. In that case

$$1 << \frac{S_E(x,t)}{\hbar}$$

with the consequence that the phase factor oscillates as a function of x and t rapidly indeed. The existence of such a wave function raises a non-trivial problem:
If the wave and its dynamical phase, and hence the wave intensity, is defined over all of space-time, how is it possible that a particle traces out a sharp and well defined path in space-time when we are left with three delemas?

1. The large magnitude ($S \gg \hbar$=1.05×10^{-27}[erg sec]) of the action for a classical particle is certainly of no help.

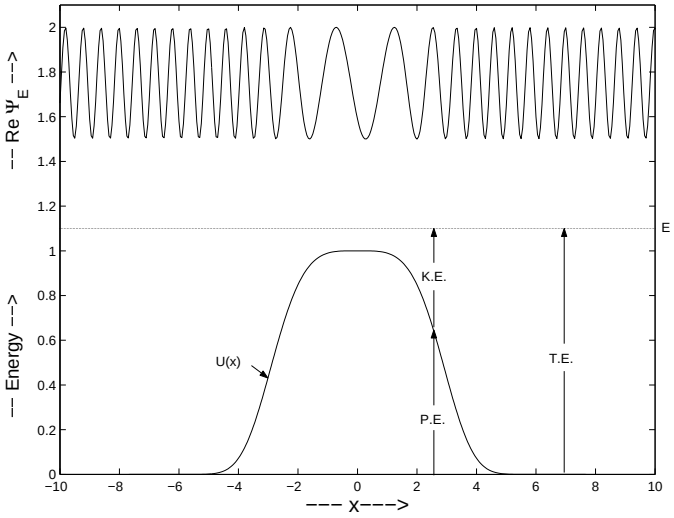


Figure 3.11: The spatial oscillation rate of the wave function $Re\,\Psi_E$ at $t = const.$ is proportional to its x-momentum, whose square is proportional to the kinetic energy (K.E.=T.E.-P.E.).

2. Neither is the simplicity of the H-J equation

$$\frac{\partial S}{\partial t} + H(x, \frac{\partial S}{\partial x}, t) = 0$$

which governs the dynamical phase in

$$\Psi = \mathcal{A} \exp\left(i \frac{S}{\hbar}\right) ,$$

3. Nor is the simplicity of the solution S for a particle of energy E,

$$S(x,t) = -Et + \int_{x_o}^{x} \sqrt{2m(E - U(x))}dx + \delta(E)$$

of any help in identifying a localized trajectory ("world line") of the particle in space-time coordinatized by x and t.

What *is* of help is the basic implication of associating a wave function with a moving particle, namely, it is a linear superposition of monochromatic waves, Eq. (3.19), which gives rise to a travelling wave packet – a localized moving wave packet whose history is the particle's world line. To validate this claim we shall give two heuristic arguements (i-ii), one application (iii), a more precise argument (iv) and an observation (v).

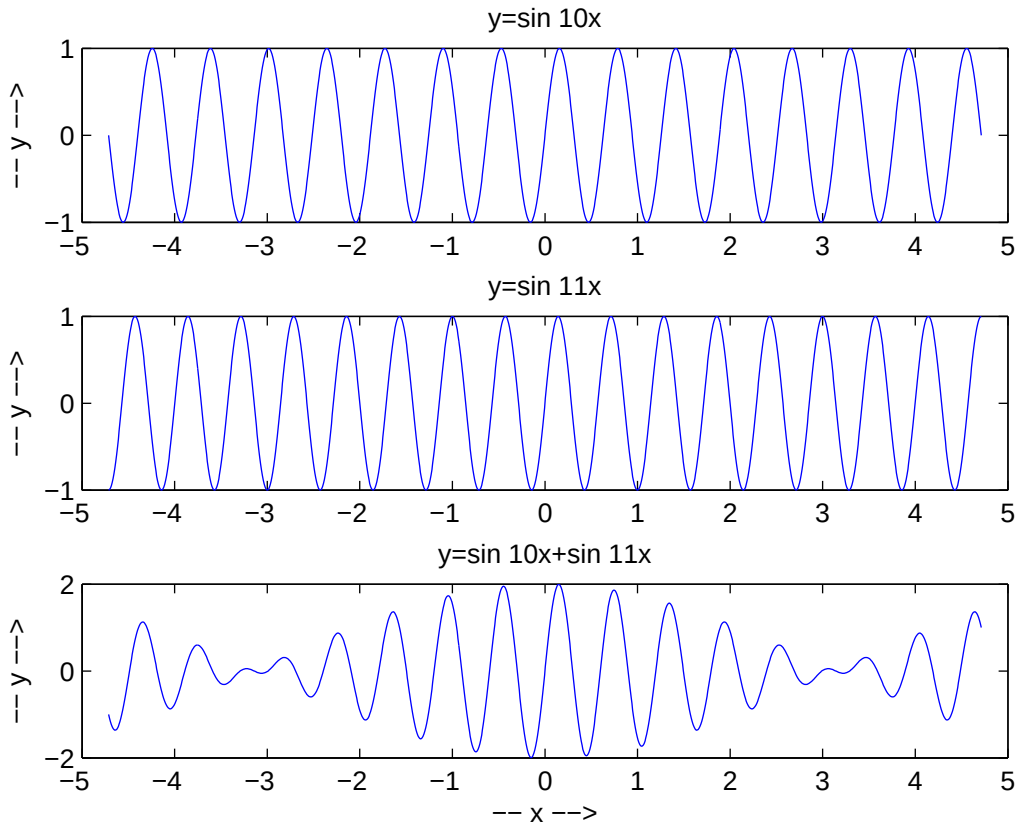


Figure 3.12: Photographic snapshot in space of two interfering wave trains and their resulting wave packet.

(i) The most elementary superposition monochomatic waves is given by the sum wave trains with different wavelengths

$$\Psi(x,t) = \Psi_E(x,t) + \Psi_{E+\Delta E}(x,t) + \cdots .$$

(ii) In space-time one has the following system of level surfaces for $S_E(x,t)$ and $S_{E+\Delta E}(x,t)$

Destructive interference between different waves comprising Ψ(x,t) occures everywhere except where the phase of the waves agree:

$$S_E(x,t) = S_{E+\Delta E}(x,t)$$

At the locus of events satisfying this condition, the waves interfere constructively and wave packet has non-zero amplitude. The quantum priciple says

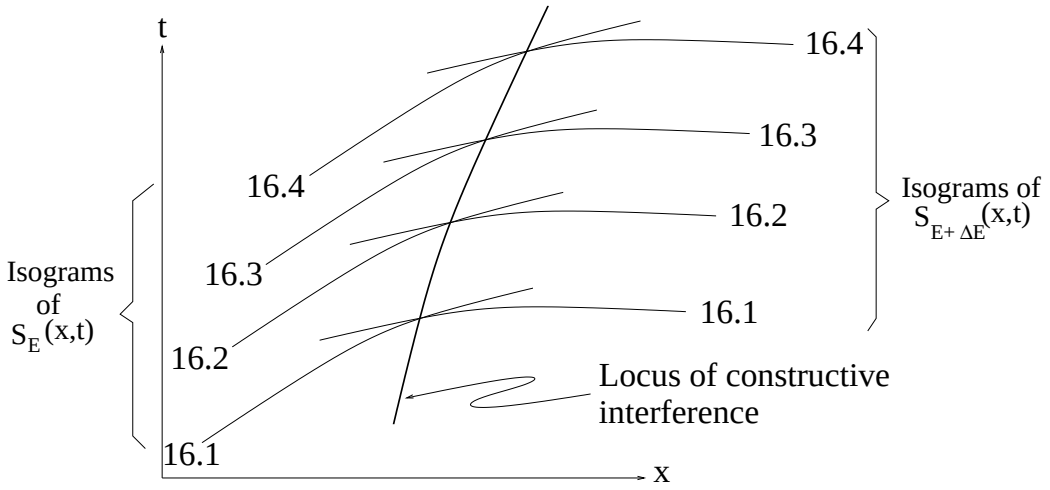


Figure 3.13: Constructive interference represented in space-time. The intersection of the respective isograms of $S_E(x,t)$ and $S_{E+\Delta E}(x,t)$ locates the events (x,t) which make up the trajectory of the particle in x-t space – the locus of constructive interference.

that this condition of constructive interference

$$0 = \lim_{\Delta E \to 0} \frac{S_{E+\Delta E}(x,t) - S_E(x,t)}{\Delta E} = \frac{\partial S_E(x,t)}{\partial E}$$

yields a Newtonian worldline, i.e. an extremal paths.

(iii) Apply this condition to the action S(x,t) of a single particle. One obtains the time the particle requires to travel to point x,

$$0 = -t + \int_{x_0}^{x} \sqrt{\frac{m}{2}} \left(\frac{1}{E - U(x)} \right)^{\frac{1}{2}} dx + t_0$$

with

$$t_0 \equiv \frac{\partial \delta(E)}{\partial E} \, .$$

This condition yields the Newtonian worldline indeed. The precise arguement is Lecture 13. The additional observation is on p13 Lecture 13.

Lecture 12

3.7 Constructive Interference

Our formulation of constructive interference is based on a picture in which at each time t a superposition of wave trains

$$\Psi_E(x,t) + \Psi_{E+\Delta E}(x,t) + \cdots \equiv \Psi(x,t)$$

yields a wave packet at time t. The principle of constructive interference itself,

$$\frac{\partial S_E(x,t)}{\partial E} = 0$$

is a condition which at each time t locates where the maximum amplitude of the wave packet is.
It is possible to bring into much sharper focus the picture of superposed wave trains and thereby not only identify the location of the resultant wave packet maximum, but also width of that packet.

3.8 Spacetime History of a Wave Packet

The sharpened formulation of this picture consists of replacing a *sum* of superposed wave amplitudes with an *integral* of wave amplitudes

$$\begin{aligned} \Psi(x,t) &= \text{"}\Psi_E(x,t) + \Psi_{E+\Delta E}(x,t) + \cdots\text{"} \\ &= \int_{-\infty}^{\infty} f(E)\, e^{\frac{i}{\hbar}S_E(x,t)} dE\,. \end{aligned} \tag{3.20}$$

A very instructive example is that of a superpostion of monochromatic ("single energy") wavetrains, each one weighted by the amplitude $f(E)$ of a Gaussian window in the Fourier ("energy") domain,

$$f(E) = Ae^{-(E-E_0)^2/\epsilon^2} \tag{3.21}$$

The dominant contribution to this integral comes from within the window, which is centered around the location of E_0 of the Gaussian maximum and has width 2ϵ, which is small for physical reasons. Consequently, it suffices to represent the phase function as a Taylor series around that central point E_0, namely

$$S_E(x,t) = S_{E_0}(x,t) + \left.\frac{\partial S_E(x,t)}{\partial E}\right|_{E_0} (E-E_0) + \frac{1}{2}\left.\frac{\partial^2 S_E(x,t)}{\partial E^2}\right|_{E_0} (E-E_o)^2 + \begin{matrix} higher \\ order \\ terms \end{matrix}\,, \tag{3.22}$$

and neglect the higher order terms. Keeping only the first three terms and ignoring the remainder allows an exact evaluation of the Gaussian superposition integral. This evaluation is based on the following formula

$$\int_{-\infty}^{\infty} e^{\alpha z^2+\beta z} dz = \sqrt{\frac{\pi}{-\alpha}} e^{-\frac{\beta^2}{4\alpha}} . \tag{3.23}$$

Applying it to the superpostion integral, Eq. (3.20) together with Eqs. (3.21) and (3.22), we make the following identification

$$\begin{aligned} z =& E - E_0 ; \quad dz = dE , \\ \alpha =& -\frac{1}{\epsilon^2} + \frac{i}{\hbar}\frac{1}{2}\left.\frac{\partial^2 S_E(x,t)}{\partial E^2}\right|_{E_0} \equiv -\frac{1}{\epsilon^2}(1-i\sigma) , \\ -\frac{1}{\alpha} =& \frac{\epsilon^2}{1-i\sigma} = \epsilon^2 \frac{1+i\sigma}{1+\sigma^2} , \\ \sigma =& \frac{1}{2}\frac{1}{\hbar}\left.\frac{\partial^2 S_E(x,t)}{\partial E^2}\right|_{E_0} \epsilon^2 , \\ \beta =& \frac{i}{\hbar}\left.\frac{\partial^2 S_E(x,t)}{\partial E^2}\right|_{E_0} . \end{aligned} \tag{3.24}$$

Inserting these expressions into the righthand side of the formula (3.23), one obtains

$$\begin{aligned} \Psi(x,t) &= A\sqrt{\pi}\epsilon\sqrt{\frac{1+i\sigma}{1+\sigma^2}} \exp\left\{ -\frac{1}{4}\left(\frac{\frac{\partial S(x,t)}{\partial E_o}}{\hbar}\right)^2 \epsilon^2 \left(\frac{1+i\sigma}{1+\sigma^2}\right)\right\} e^{i\frac{S_{E_o}(x,t)}{\hbar}} \\ &\equiv \underbrace{\mathcal{A}(x,t)}_{\substack{slowly \\ varying}} \underbrace{e^{i\frac{S_{E_o}(x,t)}{\hbar}}}_{\substack{rapidly \\ varying}} . \end{aligned}$$

This is the *rapidly oscillating* function

$$e^{i\,S_{E_0}(x,t)/\hbar}$$

modulated by a *slowly varying* amplitude $\mathcal{A}(x,t)$. For each time t this product represents a wave packet. The location of the maximum of this wave packet is given implicitly by

$$\left.\frac{\partial S_E(x,t)}{\partial E}\right|_{E_0} = 0 . \tag{3.25}$$

As t changes, the x-location of the maximum changes. Thus we have curve in x-t space of the locus of those events where the slowly varying amplitude $\mathcal{A}$ has a

maximum. In other words, this wave packet maximum condition locates those events (= points in spacetime) where constructive interference takes place.

A wave packet has finite extent in space and in time. This extent is governed by its squared modulus, i.e. the squared magnitude of its slowly varying amplitude,

$$|\Psi(x,t)|^2 = |\mathcal{A}|^2 = A^2\pi\epsilon^2 \frac{1}{\sqrt{1+\sigma^2}} \exp \underbrace{\left\{ -\frac{\epsilon^2}{2}\frac{1}{\sqrt{1+\sigma^2}} \frac{\left(\frac{\partial S_E(x,t)}{\partial E}\Big|_{E_0} \right)^2}{\hbar^2} \right\}}_{\mathbf{E}(x,t)} \tag{3.26}$$

We see that this squared amplitude has nonzero value even if the condition for constructive interference, Eq.(3.25), is violated. This violation is responsible for the finite width of the wave packet. More precisely, its shape is controlled by the exponent $\mathbf{E}(x,t)$,

$$\mathbf{E}(x,t) \equiv \left\{ -\frac{\epsilon^2}{2} \frac{1}{\sqrt{1+\left(\frac{\epsilon^2}{2\hbar} \frac{\partial^2 S_E(x,t)}{\partial E^2}\Big|_{E_0} \right)^2}} \frac{\left(\frac{\partial S_E(x,t)}{\partial E}\Big|_{E_0} \right)^2}{\hbar^2} \right\} \neq 0\,.$$

The spacetime evolution of this shape is exhibited in Figure 3.14. Thus the world-line of the particle is not a sharp one, but instead has a slight spread in space and in time. How large is this spread?

The magnitude of the wave mechanical ("non-classical") spread in the world line is the width of the Gaussian wave packet. This spread is Δx, the amount by which one has to move away from the maximum in order that the amplitude profile change by the factor $e^{\frac{1}{2}}$ from the maximum value. Let us calculate this spread under the circumstance where the effect due to dispersion is a minimum, i.e. when σ is neglegibly small. In that case the condition that $\mathbf{E}(x+\Delta x,t) = -1$ becomes

$$\left| \frac{\epsilon}{\hbar} \frac{\partial S_E(x+\Delta x,t)}{\partial E}\Big|_{E_0} \right| = 1\,.$$

Expand the left hand side to first order, make use of the fact that (x,t) is a point in spacetime where the wavepacket profile has a maximum, i.e. satisfies Eq.(3.25). One obtains

$$\left| \epsilon \frac{\partial^2 S}{\partial E \partial x} \Delta x \right| = \hbar$$

or, in light of $\partial S_E(x,t)/\partial x \equiv p(x,t;E)$,

$$\left| \epsilon \frac{\partial p}{\partial E} \Delta x \right| \;\; = \;\; \hbar\,,$$

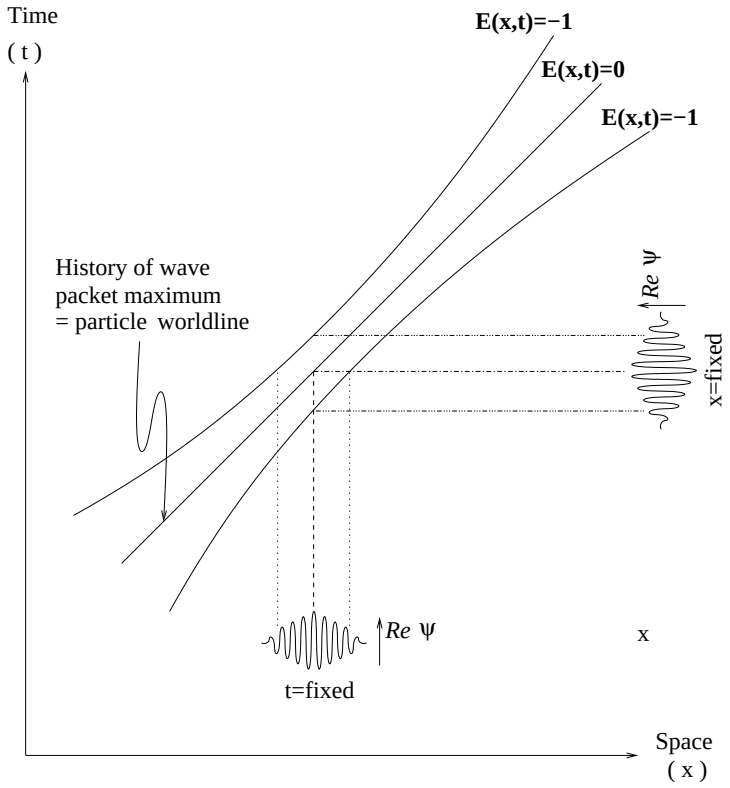


Figure 3.14: Spacetime particle trajectory ("the $\mathbf{E}(x,t) = 0$ isogram") and the dispersive wave packet amplitude histories surrounding it. The two mutually diverging ones (both characterized by $\mathbf{E}(x,t) = -1$) in this figure refer to the front and the back end of the wave packet at each instant $t = fixed$, or to the beginning and the end of the wave disturbance passing by a fixed location $x = fixed$. The particle and the wave packet maximum are moving with a velocity given by the slope of the $\mathbf{E}(x,t) = 0 = \left.\frac{\partial S_E(x,t)}{\partial E}\right|_{E_0}$ isogram, which is the locus of constructive interference exhibited in Figure 3.13

On the other hand, the convergence and subsequent divergence ("dispersion") of the wave packet is controlled (and expressed mathematically) by the behavior of the second derivative, $\left.\frac{\partial^2 S_E(x,t)}{\partial E^2}\right|_{E_0}$ of the dynamical phase $S_E(x,t)$. Whereas the behavior of its first derivative characterizes the difference in the motion of particles launched with difference initial conditions, its second derivative characterizes the intrinsically wave mechanical aspects of each of these particles.

and hence

$$\boxed{\Delta p \Delta x = \hbar}$$

Similarly the temporal extent Δt, the amount by which one has to wait (at fixed x) for the wave amplitude profile to decrease by the factor $e^{-1/2}$ from its maximum value, satisfies the condition

$$\left| \frac{\epsilon}{\hbar} \frac{\partial S_E(x, t + \Delta t)}{\partial E} \bigg|_{E_0} \right| = 1$$

which become

$$\left| \epsilon \frac{\partial^2 S_E}{\partial E \partial t} \bigg|_{E_0} \Delta t \right| = \hbar$$
$$\left| \epsilon(-) \frac{\partial E}{\partial E} \bigg|_{E_0} \Delta t \right| = \hbar$$

or

$$\boxed{\Delta E \Delta t = \hbar}\,.$$

The two boxed equation are called the Heisenberg indeterminacy relation. Even though we started with the dynamical phase S (see page 102) with $\Psi \sim e^{\frac{iS}{\hbar}}$ to arrive at the extremal path in spacetime, the constant $\hbar$ ("quantum of action") never appeared in the final result for the spacetime trajectory. The reason is that in the limit

$$\frac{S}{\hbar} \to \infty$$

the location of the wave packet reduces to the location of the wave crest. Once one knows the dynamical phase $S(x, t)$ of the system, the condition of constructive interference gives *without approximation* the location of the sharply defined Newtonian world line, the history of this wave crest, an extremal path through spacetime.

3.9 Hamilton's Equations of Motion

To validate the claim that constructive interference leads to the extremal paths determined by the E-L equations, one must first recast them into a form that involves only q^i and p_j instead of $q^i$and $\dot{q}^j$. Holding off that validation until Section 3.11, we

achieve the transition from $(q^i, \dot{q}^j)$ to (q^i, p_j) as follows:
The Lagrangian is a function of $q^i$and $\dot{q}^i$. Consequently,

$$dL = \sum_i \frac{\partial L}{\partial q^i} dq^i + \sum_i \frac{\partial L}{\partial \dot{q}^i} d\dot{q}^i$$

which may be rewritten as

$$dL = \sum_i \dot{p}_i dq^i + \sum_i p_i d\dot{q}^i \,,$$

where

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

and

$$\dot{p}_i = \frac{\partial L}{\partial q^i}$$

by the E-L equations. Using the fact that

$$p_i d\dot{q}^i = d(p_i \dot{q}^i) - \dot{q}^i dp_i \,,$$

one obtains, after a sign reversal, an expression which depends only on q^i and p_i:

$$d \underbrace{\left(\sum_i p_i \dot{q}^i - L \right)}_{H} = - \sum_i \dot{p}_i dq^i + \sum_i \dot{q}^i dp_i \,. \tag{3.27}$$

Introduce the Hamiltonian of the system

$$H(p, q, t) \equiv \sum p_i \dot{q}^i - L \,.$$

Compare its differential

$$dH = \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i + zero \times d\dot{q}^i$$

with the one given above by Eq.(3.27). Recall that two differentials are equal if and only if the coefficients of the (arbitrary) coordinate differences (i.e. $dq^1, \cdots, dq^s$, $dp_1, \cdots, dp_s$) are equal. Consequently, one has

$$\boxed{\dot{q}^i = \frac{\partial H}{\partial p_i} \,, \qquad p_i = -\frac{\partial H}{\partial q_i} \,.}$$

These are the *Hamilton's* or the *canonical equations of motion.* They are equivalent to those of E-L. Comment 1: The fact that H does not depend on $\dot{q}^i$ follows directly from

$$\frac{\partial H}{\partial \dot{q}^i} = p_i - \frac{\partial L}{\partial \dot{q}^i} = 0$$

Comment 2: a) It follows from Hamilton's equations of motion that

$$\begin{aligned} \frac{dH}{dt} &= \frac{\partial H}{\partial q^i}\dot{q}^i + \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t} \\ &= \frac{\partial H}{\partial t} \end{aligned}$$

In other words, if H is independent of any *explicit* time dependence, i.e. time then H i*s a constant* along the trajectory of the evolving system.
Comment 2: b.) If H is independent of the *generalized coordinate* q^k, then

$$\frac{dp^k}{dt} = 0$$

i.e. p_k *is a constant* for the evolving system.

3.10 The Phase Space of a Hamiltonian System

The 2s-dimentional spaced is spanned by the coordinates

$$[q^1, \cdots, q^s, p_1, \cdots, p_s]$$

is called the phase space of the system. In this phase space, the curve

$$[q^i(t), p_i(t)]$$

is an integral curve of the Hamiltonian or phase path vector field

$$\left(\frac{\partial H(q,p)}{\partial p_i}, -\frac{\partial H(q,p)}{\partial q^i}\right)$$

In other words, the tangents to this curve are given by

$$(\dot{q}^i, \dot{p}_i) = \left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q^i}\right)$$

Example: For the simple harmonic oscilator
the Lagarangian is:

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

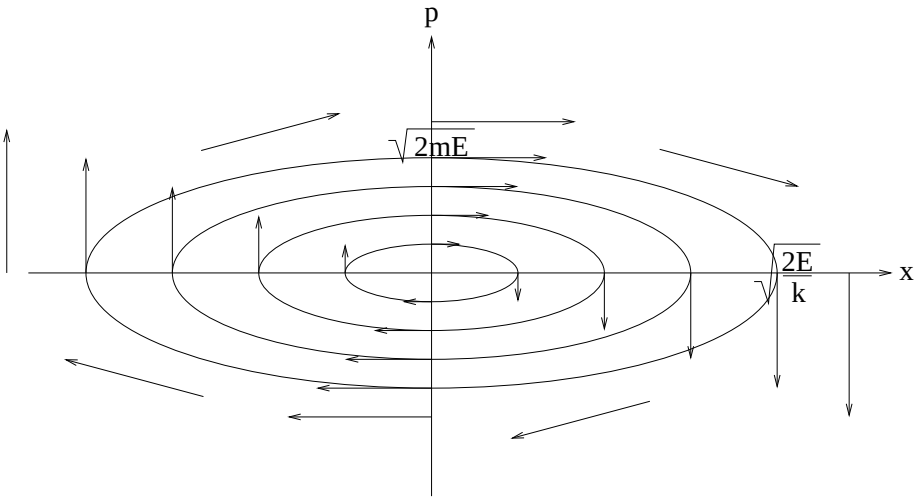


Figure 3.15: Hamiltonian vector field of a simple harmonic oscillator (s.h.o.) of mass m and spring constant k. The ellipses are integral curves whose tangents at each point are the vectors of that field. The major axis, $\sqrt{\frac{2E}{k}}$, and the minor axis, $\sqrt{2mE}$, of each ellipse are determined by the energy E of the s.h.o. The area of any particular ellipse is $2\pi\, E\, \frac{m}{k}$.

and the Hamiltonian is:

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 = E$$

a) The phase space of this system is spanned by x and p.
The Hamiltonian vector field is

$$\left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q^i}\right) = \left(\frac{p}{m}, -kx\right)$$

b) The area of one of the phase-path ellipses is

$$\text{area} = \int p dx \qquad \text{and it has the dimension of "action"}$$

According to quantum mechanics the action of a periodic system must obey the Bohr quantization condition

$$\int p dx = \left(n + \frac{1}{2}\right) h\,, \qquad n = 1, 2, \cdots \tag{3.28}$$

Thus, as depicted in Figure 3.16, the quantum mechanically allowed phase space ellipses differ in area from one another by precisely $h = 6.27\times10^{-27}$erg sec, which is

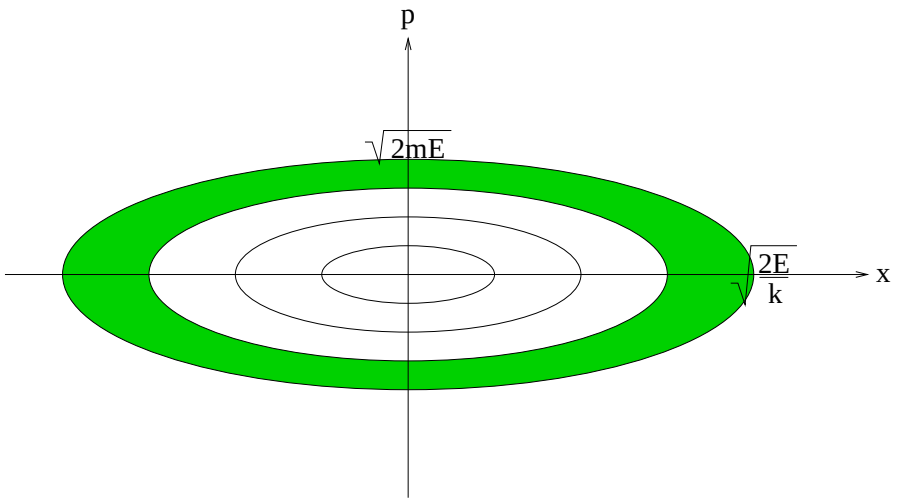


Figure 3.16: The shaded difference between the areas of adjacent phase space ellipses, Eq.(3.28), is precisely $h = 6.27\times10^{-27}$erg sec, which is one quantum of action.

one quantum of action.

For the simple harmonic oscilator the area of one of these ellipses is $\int pdx = \pi\sqrt{2mE}\sqrt{\frac{2E}{k}} = 2\pi E\sqrt{\frac{m}{k}} = 2\pi\frac{E}{\omega}$. Thus the Bohr quantization condition yields

$$2\pi\frac{E}{\omega} = \left(n + \frac{1}{2}\right)h$$

or with $\frac{\omega}{2\pi} = frequency$

$$E = (n + \frac{1}{2})h \times frequency$$

3.11 Consturctive interference ⇒ Hamilton's Equations

The principle of constructive interference provides the bridge between particle and wave mechanics. This fact is validatede by the following theorem.

Theorem. *Constructive interference conditions imply the Hamilton's equantions of motion and hence determine the existence of an extremal path.*

Proof: Step 1.) Consider a *complete integral* of the H-J equation

$$S = S(t, q^1, \cdots, q^s, \alpha_1, \cdots, \alpha_s)$$

i.e. a solution which has as many independent arbitrary constants as there are independent coordinates[4]. The constructive interference conditions are

$$\frac{\partial S}{\partial \alpha_k} = 0 \qquad k = 1, \cdots, s$$

They determine implicitly a trajectory $q^i = q^i(t),\ i = 1, \cdots, s$.
Step 2.) Take the total derivative and obtain

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial S}{\partial \alpha_k} = \frac{\partial^2 S}{\partial t \partial \alpha_k} + \frac{\partial^2 S}{\partial q^i \partial \alpha_k}\frac{dq^i}{dt} \\ &= -\frac{\partial}{\partial \alpha_k} H(q, \frac{\partial S(t,q,\alpha)}{\partial x^i}, t) + \frac{\partial^2 S}{\partial q^i \partial \alpha_k}\frac{dq^i}{dt} \\ &= -\frac{\partial H}{\partial p_i}\frac{\partial^2 S}{\partial \alpha_k \partial q^i} + \frac{\partial^2 S}{\partial q^i \partial \alpha_k}\frac{dq^i}{dt} \\ &= \frac{\partial^2 S}{\partial \alpha_k \partial q^i}\left(\frac{dq^i}{dt} - \frac{\partial H}{\partial p_i}\right), \end{aligned}$$

which implies the 1st half of Hamilton's equations,

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i},$$

provided $\frac{\partial^2 S}{\partial \alpha_k \partial q^i}$ is non-singular.
Step 3.) Differentiate both sides of the H-J equation

$$\begin{aligned} 0 &= \frac{\partial}{\partial q^i}\left[\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right)\right] \\ &= \frac{\partial}{\partial t}\frac{\partial S}{\partial q^i} + \left.\frac{\partial H}{\partial p_k}\right)_q \frac{\partial^2 S}{\partial q^k \partial q^i} + \left.\frac{\partial H}{\partial q^i}\right)_p \\ &= \frac{\partial}{\partial t}\frac{\partial S}{\partial q^i} + \frac{dq^k}{dt}\frac{\partial}{\partial q^k}\left(\frac{\partial S}{\partial q^i}\right) + \left.\frac{\partial H}{\partial q^i}\right)_p \\ &= \frac{d}{dt}p_i + \frac{\partial H}{\partial q^i} \end{aligned}$$

[4]Such independence is expressed mathematically by the fact that

$$det\left|\frac{\partial^2 S}{\partial q^i \partial \alpha_j}\right| \neq 0\,.$$

This condition would be violated if the dependence on two constants were of the form $S(t, q^i, f(\alpha_1, \alpha_2), \alpha_3, \cdots, \alpha_s)$.

which is the 2nd half of Hamilton's equation's,

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \, .$$

QED. Thus the two Hamilton's equations of motion are implied by the principle of constructive interference indeed.

Lecture 13

3.12 Applications

Two of the most important applications of Hamilton-Jacobi theory are found in

(i) the motion of bodies on the astronomical scale, for example, space craft, comets, or planets moving in a gravitational field, and in

(ii) the motion of bodies on the atomic scale, for example, a charged particle (electron) moving in the potential of an atomic nucleus or in th electromagnetic field of a pulsed laser.

The mathematical procedure for these and numerous other examples is routine and always the same:
(i) Construct the Hamiltonian for the system
(ii) Write down and solve the H-J equation
(iii) Apply the conditions of constructive interference to obtain the trajectories of the body.
Let us describe how this three step procedure is done in practice.

3.12.1 H-J Equation Relative to Curvilinear Coordinates

In constructing the Hamiltonian one must choose some specific set of coordinates. For a single particle it is is difficult to find an easier way of writing down the H-J equation than the way whose starting point is the element of arclength

$$\begin{aligned} (ds)^2 &= dx^2 + dy^2 + dz^2 \quad \text{(Cartesian coordinates)} \\ &= g_{ij} dx^i dx^j \qquad \text{(curvilinear coordinates)} \end{aligned} \tag{3.29}$$

This element of arclength is the best starting point because it is related so closely to the Lagrangian of the system. Indeed, one has

$$\begin{aligned} L &= \frac{1}{2} m \vec{x} \cdot \vec{x} - U \\ &= \frac{1}{2} m \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right] - U \\ &= \frac{1}{2} m \, g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} - U \, . \end{aligned}$$

In other words, the Lagrangian is constructed relative to curvilinear coordinates by inspection. The steps leading to the H-J equation are now quite routine.

The momenta are

$$p_j = \frac{\partial L}{\partial \dot{x}^j} = m \, g_{ij} \dot{x}^i \, .$$

Let g^{kj}be the inverse of g_{ji}: $g^{kj} g_{ji} = \delta^k_i$ so that

$$\dot{x}^i = \frac{1}{m} \, g^{ij} p_j$$

and

$$H = p_j \dot{x}^j - L = \frac{1}{2m} \, g^{ij} p_i p_j + U$$

Thus the Hamilton-Jacobi equation is

$$-\frac{\partial S}{\partial t} = \frac{1}{2m} \, g^{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} + U$$

in terms of the inverse metric.

3.12.2 Separation of Variables

The most important way of solving the H-J equation is by the method of sep[aration of variables. To illustrate this, consider the following

Example (Particle in a dipole potential) Consider the motion of a particle in the potential of a combined dipole and monopole field. Relative to spherical coordinates the metric is

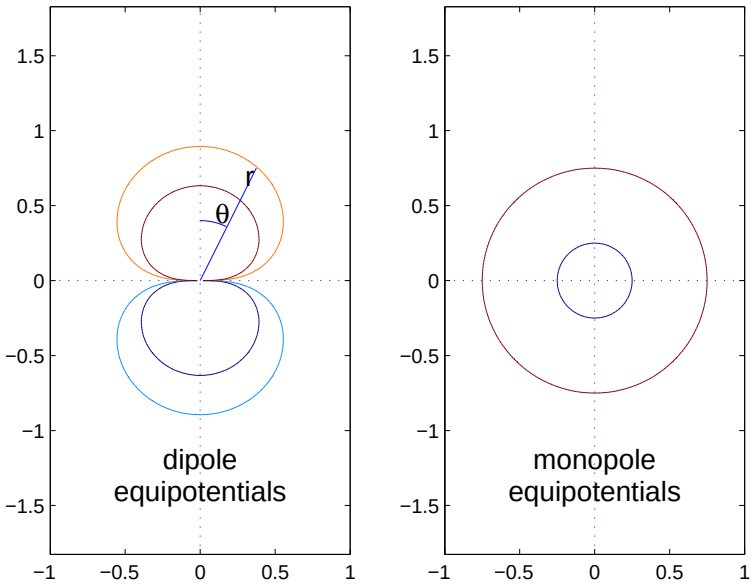


Figure 3.17: Rotationally symmetric potential as the sum a dipole potential $\left(\mu\frac{\cos\theta}{r^2}\right)$ plus a monopole potential $\left(-\frac{k}{r}\right)$.

$$(ds)^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

and that potential has the form

$$U(r,\theta) = \mu\frac{\cos\theta}{r^2} - \frac{k}{r}$$

Its equipotential surfaces are rotationally symmetric around the z-axis

The Lagrangian is

$$\begin{aligned} L &= \text{Kinetic Energy } - \text{Potential Energy} \\ &= \frac{1}{2} m\, g_{ij}\dot{q}^i\dot{q}^j - U(q^1,q^2,q^3) \\ &= \frac{1}{2} m\,(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - \mu\frac{\cos\theta}{r^2} + \frac{k}{r}\,. \end{aligned}$$

The corresponding Hamilton-Jacobi equation is

$$
\begin{aligned}
0 &= \frac{\partial S}{\partial t} + H \\
&= \frac{\partial S}{\partial t} + \frac{1}{2m} g^{ij} \frac{\partial S}{\partial q^i} \frac{\partial S}{\partial q^j} + U(q^1, q^2, q^3) \\
&= \frac{\partial S}{\partial t} + \frac{1}{2m} \left[\left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial S}{\partial \phi} \right)^2 \right] + \mu \frac{\cos \theta}{r^2} - \frac{k}{r} \,.
\end{aligned}
$$

This equation can be solved by the method of separation of variables. This method is condensed into the following three definitions and propositions:

1. *Definition (Separable variables).* The variables $q^1, q^2, \cdots, q^s$ in the H-J equation

$$
0 = \frac{\partial S}{\partial t} + H \left(t, \, q^i, \frac{\partial S}{\partial q^j} \right) \equiv \mathcal{H} \left(t, \frac{\partial S}{\partial t}, q^i, \frac{\partial S}{\partial q^j} \right)
$$

are said to be *separable* if it has a "complete" solution of the form

$$
S = S_0(t, \alpha_0) + S_1(q^1, \alpha_0, \alpha_1) + S_2(q^2, \alpha_0, \alpha_1, \alpha_2) + S_3(q^3, \alpha_0, \alpha_1, \alpha_2, \alpha_3) \tag{3.30}
$$

where each S_i depends only on t and q^i respectively.

2. *Definition (Complete solution).* A solution is said to be *complete* if

$$
\det \left| \frac{\partial^2 S}{\partial \alpha_i \partial q^j} \right| \neq 0
$$

Remark 1: We saw (in Section 3.11 on page 128) in the context of reconstructing the classical trajectories of a Hamiltonian system from the principle of constructive interference it was essential that the matrix $[\partial^2 S / \partial \alpha_i \partial q^j]$ be non-singular.
Remark 2: The solution, Eq.(3.30), is complete indeed, because

$$
\det \left| \tfrac{\partial^2 S}{\partial \alpha_i \partial q^j} \right| = \det \begin{array}{|ccc|c|c} 1 & 2 & 3 & \leftarrow & j \\ * & 0 & 0 & & 0 \\ * & * & 0 & \neq 0 & 1 \\ * & * & * & & 2 \\ & & & & \uparrow \\ & & & & i \end{array}
$$

and its diagonal elements are not zero.

3. *Definition (Separability condition).* The Hamilton-Jacobi equation is said to satisfy the separability criterion if its Hamiltonianis is of the form

$$\mathcal{H}\left(t, \frac{\partial S}{\partial t}, q^i, \frac{\partial S}{\partial q^j}\right) = f_3\left(f_2\left(f_1\left(f_0\left(t, \frac{\partial S}{\partial t}\right), q_1, \frac{\partial S}{\partial q^1}\right), q^2, \frac{\partial S}{\partial q^2}\right), q^3, \frac{\partial S}{\partial q^3}\right)$$

(for $s = 3$ degrees of freedom). This functional form is said satisfy the condition of separability because the solution to this first order p.d.e. has the separated form, Eq.(3.30). In other words, the claim is

Proposition: Definition 3 implies Definition 1.
Proof: Step 1.) According to Definition 3 the H-J equation is

$$f_3\left(f_2\left(f_1\left(f_0\left(t, \frac{\partial S}{\partial t}\right), q_1, \frac{\partial S}{\partial q^1}\right), q^2, \frac{\partial S}{\partial q^2}\right), q^3, \frac{\partial S}{\partial q^3}\right) = 0\,. \tag{3.31}$$

The method of solution via separation of variables starts by solving for f_0. One finds

$$f_0\left(t, \frac{\partial S}{\partial t}\right) = \text{an expression involving } q^1, q^2, q^3, \frac{\partial S}{\partial q^1}, \frac{\partial S}{\partial q^2}, \text{ and } \frac{\partial S}{\partial q^3} \text{ only }.$$

It follows that the solution must have the separated form

$$S = T(t) + S'(q^1, q^2, q^3)\,. \tag{3.32}$$

This is because the common value of

$$\underbrace{f_0\left(t, \frac{dT(t)}{dt}\right)}_{\substack{\text{independent} \\ \text{of } q^1, q^2, q^3}} = \underbrace{\text{an expression that depends only on } q^1, q^2, q^3}_{\text{independent of t}}$$

is independent of all variables. This common independence implies that f_0 is a constant, say, α_0:

$$f_0\left(t, \frac{dT(t)}{dt}\right) = \alpha_0\,. \tag{3.33}$$

Solving for $T(t)$, one obtains

$$T(t) = S_0(t, \alpha_0)\,. \tag{3.34}$$

and, in light of Eq(3.32),

$$\frac{dT}{dt} = \frac{\partial S}{\partial t} \tag{3.35}$$

so that

$$f_0\left(t, \frac{\partial S}{\partial t}\right) = \alpha_0\,. \tag{3.36}$$

Step 2.) Introduce this function value into Eq.(3.31). This remove its t-dependence. The simplified functional equation is

$$f_3\left(f_2\left(f_1\left(\alpha_0, q^1, \frac{\partial S'}{\partial q^1}\right), q^2, \frac{\partial S'}{\partial q^2}\right), q^3, \frac{\partial S'}{\partial q^3}\right) = 0\,.$$

Now solve for f_1 and find that

$$f_1\left(\alpha_0, q^1, \frac{\partial S'}{\partial q^1}\right) = \text{an expression involving } \; q^2, q^3, \frac{\partial S'}{\partial q^2}, \text{and} \frac{\partial S'}{\partial q^3} \; \text{only.}$$

It follows that the solution S' has the separated form

$$S' = Q_1(q^1) + S''(q^2, q^3)\,. \tag{3.37}$$

This is because

$$\underbrace{f_1\left(\alpha_0, q^1, \frac{dQ_1(q^1)}{dq^1}\right)}_{\substack{\text{independent} \\ \text{of } q^2, q^3}} = \underbrace{\text{an expression that depends only on } \; \text{on}\, q^2, q^3}_{\text{independent of q}^1}\,.$$

This common independence implies that f_1 is a constant, say, α_1:

$$f_1\left(\alpha_0, q^1, \frac{dQ_1(q^1)}{dq^1}\right) = \alpha_1\,.$$

Solving for $Q_1(q^1)$, one obtains

$$Q_1(q^1) = S_1(q^1, \alpha_1, \alpha_0)\,. \tag{3.38}$$

and, in light of Eq.(3.37),

$$\frac{dQ_1}{dq^1} = \frac{\partial S'}{\partial q^1} \tag{3.39}$$

At this stage one has brought the solution, Eq.(3.32), to the H-J equation has the into the separated form

$$S = T(t) + Q_1(q^1) + S''(q^2, q^3)\,.$$

Step 3.) Repeat Step 2.) two more times to obtain

$$f_2\left(\alpha_0, \alpha_1, q^2, \frac{dQ_2(q^2)}{dq^2}\right) = \alpha_2$$
$$f_3\left(\alpha_0, \alpha_1, \alpha_2, q^3, \frac{dQ_3(q^3)}{dq^{23}}\right) = \alpha_3\,.$$

Notice, however, that the H-J Eq.(3.31) implies that $\alpha_3 = 0$, always. Consequently, there are only *three* independent separation constants, $(\alpha_0, \alpha_1, \alpha_2)$, while the number of independent variables, (t, q^1, q^2, q^3) is four. It follows that

$$S = T(t) + Q_1(q^1) + Q_2(q^2) + Q_3(q^3)$$

and hence with Eqs.(3.34), (3.38), $\cdots$etc. The to-be-determined solution S appears in the H-J euation only in the form of its derivatives. Consequently, S necessarily contains an additive constant, which in general depends on the three separation constants $\alpha_0, \alpha_1, \alpha_3$. Thus the dynamical phase S has indeed the separated form whenever its H-J equation has the form (3.31):

$$S = S_0(t, \alpha_0) + S_1(t, q^1, \alpha_0, \alpha_1) \\ + S_2(t, q^2, \alpha_0, \alpha_1, \alpha_2) + S_3(t, q^3, \alpha_0, \alpha_1, \alpha_2, \alpha_3 = 0) + \delta(\alpha_0, \alpha_1, \alpha_2)\,.$$

The purpose and the specific values of $\alpha_0, \alpha_1, \alpha_3$ and the form of δ mathematize the specific nature of the particular system under consideration.

Lecture 14

Motion in the Field of Two Unequal Bodies: Prolate Spheroidal Coordinates

As an application of Hamilton-Jacobi theory, consider the problem of determining the motion of a particle under the influence of forces toward two fixed centers.

The force due to each center is directed radially and proportional to the inverse squared distance.

[FIGURE]

The potential is therefore

$$U = \frac{\kappa}{r} + \frac{\kappa'}{r'} = \frac{\kappa}{\sqrt{\rho^2 + (z-c)^2}} + \frac{\kappa'}{\sqrt{\rho^2 + (z+c)^2}}.$$

Here $2c$ is the distance between the two centers and κ_1 and κ_2 are their respective sources of the force fields. The problem of an otherwise free particle can be solved completely for such a potential.

This potential has various limiting forms, for which the dynamics of the particle can therefore also be solved exactly. They are:

A. Particle in the *field of a dipole*:

$$\begin{array}{c} \kappa = -\kappa' \\ \kappa \to \infty \\ c \to 0 \end{array} \quad \text{but} \quad c\kappa = \text{const} = \frac{\mu}{2}$$

[FIGURE]

In that case

$$U \to \mu \frac{\cos\theta}{r^2}$$

B. Particle in the field of an isolated body immersed in a uniform field. For example, the hydrogen atom in a uniform electric field.

[FIGURE]

For this case we arrange the attractive centers so that

[FIGURE]

$$\begin{array}{c} c \to \infty \\ \kappa' \to \infty \\ c \to 0 \end{array} \quad \text{but} \quad \frac{\kappa'}{c^2} = \text{const} = 4F$$

In that case

$$U \to \frac{\kappa}{\sqrt{\rho^2 + \bar{z}^2}} - F\bar{z}$$

The first task is to establish appropriate coordinates. The rotational symmetry around the z-axis suggests that we start out with cylindrical coordinates relative to the axis containing the sources, in which the squared element of arclength ("metric") has the form

$$ds^2 = g_{ij}dx^i dx^j = d\rho^2 + dz^2 + \rho^2 d\phi^2.$$

It is sufficient to focus on a plane $\phi = \text{const}$.

Instead of letting the coordinate surfaces be $\rho = \text{const}$ and $z = \text{const}$, we introduce elliptical and hyperbolic surfaces of revolution, with the force centers as their common foci.

[FIGURE]

Figure 3.18: Prolate spherioidal coordinates.

Such a set of orthogonal coordinate surfaces is most easily introduced by means of the conformal transformation

$$z + i\rho \equiv w = f(\rho) = c\cosh q = c\cosh(q_1 + iq_2)$$

[FIGURE]

Figure 3.19: complex $q = q_1 + iq_2$ plane $\xrightarrow{f}$ complex w plane

The real and imaginary parts of this transformation are

$$\begin{aligned} w = z + i\rho &= c\cosh(q_1 + iq_2) \\ &= c\left(\cosh q_1 \cos q_2 + i \sinh q_1 \sin q_2\right) \\ z &= c\cosh q_1 \cos q_2 \\ \rho &= c\sinh q_1 \sin q_2 \end{aligned}$$

so that the locus of constant q_1 and q_2 yields

$$\begin{array}{ll} \text{ellipses} & \dfrac{z^2}{c^2\cosh^2 q_1} + \dfrac{\rho^2}{c^2\sinh^2 q_1} = 1 \qquad 0 \le q_1 < \infty \\ \text{and} & \\ \text{hyperbolas.} & \dfrac{z^2}{c^2\cos^2 q_2} - \dfrac{\rho^2}{c^2\sin^2 q_2} = 1 \qquad 0 \le q_2 < 2\pi \end{array}$$

The coordinates q_1 and q_2 together with ϕ are called prolate spheroidal coordinates.

The squared element of arclength relative to these prolate ellipsoidal coordinates is

$$\begin{aligned} ds^2 &= dz^2 + d\rho^2 + \rho^2 d\phi^2 = dw\, d\bar{w} + \rho^2 d\phi^2 \\ {\scriptstyle w=f(q)} \quad &= \underbrace{f'(q)\overline{f'(q)}}_{c^2|\sinh q|^2} \underbrace{(dq_1^2 + dq_2^2)}_{dq\, d\bar{q}} + \rho^2 d\phi^2. \end{aligned}$$

Here

$$\begin{aligned} |f'(q)|^2 = c^2|\sinh q|^2 &= c^2\big[\underbrace{\sinh^2 q_1}_{\cosh^2 q_1 - 1}\cos^2 q_2 + \cosh^2 q_1 \underbrace{\sin^2 q_2}_{1-\cos^2 q_2}\big] \\ &= c^2\left[\cosh^2 q_1 - \cos^2 q_2\right] \end{aligned}$$

Thus the metric is $(ds)^2 = c^2\left(\cosh^2 q_1 - \cos^2 q_2\right)\left(dq_1^2 + dq_2^2\right) + \rho^2 d\phi^2$ where $\rho^2 = c^2 \sinh^2 q_1 \sin^2 q_2$.

The two squared radii are

$$\begin{aligned}\rho^2 + (z \pm c)^2 &= c^2 \sin^2 q_2 \sinh^2 q_1 + c^2 \left(\cosh q_1 \cos q_2 \pm 1\right)^2 \\ &= c^2 \left[\sin^2 q_2 \sinh^2 q_1 + \cosh^2 q_1 \cos^2 q_2 \pm 2 \cosh q_1 \cos q_2 + 1\right] \\ &= c^2 \left[\left(1 - \cos^2 q_2\right)\left(\cosh^2 q_1 - 1\right) + \text{etc.}\right] \\ &= c^2 \left[\cosh^2 q_1 + \cos^2 q_2 \pm 2 \cosh q_1 \cos q_2\right] \\ &= c^2 \left(\cosh^2 q_1 \pm \cos^2 q_2\right)^2\end{aligned}$$

Thus the potential $U = \frac{\kappa}{\sqrt{\rho^2+(z-c)^2}} + \frac{\kappa'}{\sqrt{\rho^2+(z+c)^2}}$ is

$$U(q_1, q_2) = \frac{\kappa}{c} \frac{1}{\cosh^2 q_1 - \cos^2 q_2} + \frac{\kappa'}{c} \frac{1}{\cosh^2 q_1 + \cos^2 q_2}.$$

The Lagrangian is

$$L = \frac{m}{2} c^2 \left[\left(\cosh^2 q_1 - \cos^2 q_2\right)^2 \left(\dot{q}_1^2 + \dot{q}_1^2\right) + \sinh^2 q_1 \sin^2 q_2 \, \dot{\phi}^2\right] - U(q_1, q_2).$$

The generalized momenta are

$$p_i = \frac{m}{2} c^2 \left(\cosh^2 q_1 - \cos^2 q_2\right) \dot{q}_i^2 \qquad i = 1, 2$$

and

$$p_\phi = m\rho^2 \dot{\phi}.$$

The Hamiltonian is

$$H = \frac{1}{2mc^2}\left[\frac{p_1^2 + p_2^2}{\cosh^2 q_1 - \cos^2 q_2} + \frac{p_\phi^2}{\left(\cosh^2 q_1 - 1\right)\left(1 - \cos^2 q_2\right)}\right] + U\left(q_1, q_2\right).$$

Note: t and ϕ are cyclic coordinates.

The Hamilton-Jacobi equation is therefore

$$\begin{aligned}E = &\frac{1}{2mc^2\left(\cosh^2 q_1 - \cos^2 q_2\right)} \times \\ &\left\{\left(\frac{\partial S_0}{\partial q_1}\right)^2 + \left(\frac{\partial S_0}{\partial q_2}\right)^2 + \left[\frac{1}{\cosh^2 q_1 - 1} + \frac{1}{1 - \cos^2 q_2}\right]\left(\frac{\partial S_0}{\partial \phi}\right)^2\right\} \\ &+ \frac{\kappa}{c}\frac{1}{\cosh^2 q_1 - \cos^2 q_2} + \frac{\kappa'}{c}\frac{1}{\cosh^2 q_1 + \cos^2 q_2}.\end{aligned}$$

One sees (after multiplying through by $\left(\cosh^2 q_1 + \cos^2 q_2\right)$) that the general solution to this equation is

$$S = -Et + S_1(q_1) + S_2(q_2) + p_\phi \phi$$

where

$$\alpha_1 + 2mc^2 E \cosh^2 q_1 - \frac{p_\phi}{\cosh^2 q_1 - 1} - 2mc(\kappa + \kappa') \cosh q_1 = \left(\frac{\partial S_1}{\partial q_1}\right)^2$$

and

$$-\alpha_1 - 2mc^2 E \cos^2 q_2 - \frac{p_\phi}{1 - \cos^2 q_2} - 2mc(\kappa - \kappa') \cos q_2 = \left(\frac{\partial S_2}{\partial q_2}\right)^2$$

so that

$$\begin{aligned} S = & -Et + p_\phi \phi \\ = & \int q_1 dq_1 \left[\alpha_1 + 2mc^2 E \cosh^2 q_1 - \frac{p_\phi}{\sinh^2 q_1} - 2mc(\kappa + \kappa') \cosh q_1\right]^{1/2} \\ & + \int q_2 dq_2 \left[-\alpha_1 - 2mc^2 E \cos^2 q_2 - \frac{p_\phi}{\sin^2 q_2} - 2mc(\kappa - \kappa') \cos q_2\right]^{1/2} \\ & + \delta(E, p_\phi, \alpha_1). \end{aligned}$$

The orbits are determined by the constructive interference conditions

$$\frac{\partial S}{\partial E} = 0, \qquad \frac{\partial S}{\partial p_\phi} = 0, \qquad \frac{\partial S}{\partial \alpha_1} = 0.$$

Recapitulation The effect of the transformation

$$z + i\rho \equiv f(\rho) = c \cosh\left(q_1 + i q_2\right) = c\left(\cosh q_1 \cos q_2 + i \sinh q_1 \sin q_2\right)$$

is to preserve angles, and in particular orthogonality

$$(ds)^2 = \overset{\text{cylindrical coord.}}{dz^2 + d\rho^2 + \rho^2 d\phi^2} = c^2 (\overset{\text{oblate spherical coord.}}{\cosh^2 q_1 - \cos^2 q_2})(dq_1^2 + dq_2^2) + \rho^2 d\phi^2.$$

[FIGURE]

The inverse metric has a form

$$\begin{aligned} (\vec{\nabla} S) \cdot (\vec{\nabla} S) &= g^{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} \\ &= \frac{1}{c^2 \left(\cosh^2 q_1 - \cos^2 q_2\right)} \left[\left(\frac{\partial S}{\partial q_1}\right)^2 + \left(\frac{\partial S}{\partial q_2}\right)^2\right] + \frac{1}{\rho^2} \left(\frac{\partial S}{\partial z}\right)^2 \end{aligned}$$

which yields the potential

$$\begin{aligned} U &= \frac{\kappa}{\sqrt{\rho^2+(z-c)^2}} + \frac{\kappa'}{\sqrt{\rho^2+(z+c)^2}} \\ &= \frac{\kappa}{c}\frac{1}{\cosh^2 q_1 - \cos^2 q_2} + \frac{\kappa'}{c}\frac{1}{\cosh^2 q_1 + \cos^2 q_2}, \end{aligned}$$

where

$$\rho^2 = c^2 \sinh q_1 \sin q_2$$
$$\rho^2 + (z \pm c)^2 = c^2 \left(\cosh^2 q_1 \pm \cos^2 q_2\right)^2$$

From these one then sets up and solves the H-J equation as done in the previous lecture.

How Spheroidal Coordinates can be Deformed into Spherical Coordinates
Consider the potential due to two attractive centers.

[FIGURE]

Consider the case $\kappa = -\kappa'$. Now let $c \to 0$, $\kappa \to \infty$ so that

$$2c\kappa = \text{const} = \mu.$$

The two prolate spheroidal coordinates q_1 and q_2, which are determined by

$$\rho = c \sinh q_1 \sin q_2$$
$$z = c \cosh q_1 \cos q_2$$

become the spherical coordinates θ and r. This is so because for fixed ρ and z ("same point in space") $c \sinh q_1$ and $c \cosh q_1$ remain fixed. Thus as $c \to 0$, $q_1 \to \infty$. Consequently

$$\left.\begin{matrix} c \sinh q_1 \\ c \cosh q_2 \end{matrix}\right\} \to \text{same constant} \equiv r$$

Thus, setting $q_1 \equiv \theta$, one obtains

$$\rho = r \sin\theta$$
$$z = r \cos\theta$$

[FIGURE]

In other words, the prolate spheroidal coordinates become the spherical coordinates.

You can convince yourself that

$$\lim_{\substack{c\to 0\\ 2\kappa c=\mu}} \underbrace{\frac{\kappa}{\sqrt{\rho^2+(z-c)^2}}-\frac{\kappa}{\sqrt{\rho^2+(z+c)^2}}}_{U} = \frac{2\kappa c z}{\sqrt{\rho^2+z^2}} = \mu\frac{\cos\theta}{r^2}.$$

In other words, the potential becomes

$$\boxed{U=\mu\frac{\cos\theta}{r^2}},$$

that of a dipole of strength μ. The H-J equation separates for this potential, and the trajectories can therefore be determined.

The limit of the potential U is

$$\begin{aligned} U &= \frac{\kappa}{\sqrt{\rho^2+(z-c)^2}}-\frac{\kappa}{\sqrt{\rho^2+(z+c)^2}} \\ &= \frac{\kappa\left[\sqrt{\rho^2+z^2+2zc+c^2}-\sqrt{\rho^2+z^2-2zc+c^2}\right]}{\sqrt{\left(\rho^2+z^2+c^2\right)^2-(2cz)^2}}. \end{aligned}$$

As $c\to 0$ the difference in the brackets goes to 0. To preserve accuracy, we must expand around p^2+z^2. We obtain

$$U = \kappa\left(\rho^2+z^2\right)^{1/2}\frac{\left[1+\frac{zc}{\rho^2+r^2}+\cdots-\left(1-\frac{zc}{\rho^2+r^2}+\cdots\right)\right]}{\left(\rho^2+z^2\right)\left\{1-\frac{2c^2z^2}{(\rho^2+z^2)^2}+\cdots\right\}}$$

$$\therefore \quad \lim_{\substack{c\to 0\\ 2\kappa c=\mu}} U = \frac{2\kappa c z}{\left(\rho^2+z^2\right)^{1/2}} = \mu\frac{\cos\theta}{r^2}$$

This is the potential of a dipole whose dipole strength is μ.

[FIGURE]

Figure 3.20: Potential of a field whose strength is $\mu = 2\kappa c$.

One can solve exactly the problem of a particle moving in such a field. The Lagrangian is

$$L = \frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2\right)-\mu\frac{\cos\theta}{r^2}.$$

The Hamilton-Jacobi equation is

$$0 = \frac{\partial S}{\partial t}+\frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial S}{\partial\theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial\phi}\right)^2-\frac{\mu}{r^2}\cos\theta.$$

This equation can be solved using the separation of variables technique:

$$S = -Et + R(r, E, \ell, p_\phi) + \Theta(\theta, \ell, p_\phi) + p_\phi \phi + \delta(E, \ell, p_\phi).$$

Here

$$\alpha_1 = E \qquad \alpha_2 = \ell \qquad \alpha_3 = p_\phi$$

are the separation constants. The orbits are given by the constructive interference condition

$$\frac{\partial S}{\partial E} = 0, \qquad \frac{\partial S}{\partial \ell} = 0, \qquad \frac{\partial S}{\partial p_\phi} = 0.$$

Lecture 15

3.13 Hamilton's Principle for the Mechanics of a Continuum

If a principle is to be gauged by the *scope* of the phenomena and processes it brings into our grasp and by the *brevity* of its annunciation, then the extremum principles of physics and engineering do not have many rivals.

Most prominent among them is Hamilton's principle, which is based on the action integral

$$I = \int_{t_0}^{t_1} (\text{K.E.} - \text{P.E.}) dt.$$

(K.E.=Kinetic Energy, P.E.=Potential Energy)

We would like to extend its range of applicability from systems having a finite number of degrees of freedom to systems having an infinite number.

Hamilton's principle accommodates such an extension, provided (i) we can identify the P.E. and the K.E. of such systems and then (ii) give a mathematical formulation for optimizing their performance as ascertained by their action integral.

Examples. (i) The P.E. of a string is the finite sum of the potential energy of its atoms. However, we shall approximate this discrete and finite system by a continuum whose state is specified by the transverse displacement $\psi(x)$. If the string is under tension T [force], then its potential energy is the energy necessary to bring it from its unstretched to the stretched state associated with its transverse displacement $\psi(x)$. The stretch energy is

$$\begin{aligned} \text{P.E.} &= \int T \left[\sqrt{1 + \left(\frac{d\psi}{dx} \right)^2} - 1 \right] dx \\ &\cong \frac{1}{2} \int T \left(\frac{d\psi}{dx} \right)^2 dx \quad \text{(small amplitude approximation)} \end{aligned}$$

(ii) If the string is embedded in an elastic medium whose elastic constant is κ and if the string is subjected to a given transverse force density $F(x)$ $[\frac{\text{force}}{\text{length}}]$, then the potential energy of the string is

$$\text{P.E.} = \int \left[\frac{1}{2} T \left(\frac{d\psi}{dx} \right)^2 + \frac{1}{2} \kappa \psi^2 - F\psi \right] dx$$

(iii) Suppose we have a soap bubble under surface tension T $[\frac{\text{force}}{\text{length}} = \frac{\text{energy}}{\text{area}}]$. This means that the energy necessary to displace a unit column (of, say, length ℓ) of soap molecules horizontally by an amount Δx is

$$\begin{aligned} \Delta \,\text{energy} &= \text{Force } \Delta x = (T\ell)\Delta x = T(\ell \Delta x) \\ &= T(\text{augmented area}) \end{aligned}$$

[FIGURE]

If this soap bubble is in a state where its transverse deformation from equilibrium is $\psi(x, y)$, then the potential energy associated with the concomitant increase in area is

$$\begin{aligned} \text{P.E.} &= \iint T \left[\sqrt{1 + \left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2} - 1 \right] dx\, dy \\ &\cong \frac{1}{2} \iint T \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right] dx\, dy \end{aligned}$$

(iv) Suppose we have a rod or a column whose Young's modulus is Y $[\frac{\text{force}}{\text{area}}]$. The

[FIGURE]

potential energy when the rod is in a compressed (or stretched) state is

$$\text{P.E.} = A\, \frac{1}{2} \int Y \left(\frac{\partial \psi}{\partial x} \right)^2 dx$$

(v) The kinetic energy in all these examples has the form

$$\begin{aligned} &\text{K.E.} = \tfrac{1}{2} \textstyle\int \rho \left(\frac{\partial \psi}{\partial t} \right)^2 dx && \rho = \left[\frac{\text{mass}}{\text{length}} \right] \\ &\quad \text{or} \\ &\text{K.E.} = \tfrac{1}{2} \textstyle\iint \rho \left(\frac{\partial \psi}{\partial t} \right)^2 dx\, dy && \rho = \left[\frac{\text{mass}}{\text{area}} \right] \\ &\quad \text{or} \\ &\text{K.E.} = \tfrac{1}{2} \textstyle\iiint \rho \left(\frac{\partial \psi}{\partial t} \right)^2 dx\, dy\, dz && \rho = \left[\frac{\text{mass}}{\text{volume}} \right] \end{aligned}$$

(vi) The typical Lagrangian for a three-dimensional medium undergoing only small deformations has the form

$$L = \text{K.E.} - \text{P.E.} = \iiint \mathcal{L}\, dx\, dy\, dz$$

where

$$\mathcal{L} = \frac{1}{2}\rho\left(\frac{\partial\psi}{\partial t}\right)^2 - \frac{1}{2}k\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right] - \frac{\kappa}{2}\psi^2 + F\psi$$

is the *Lagrangian density* of the system.

3.13.1 Variational Principle

Consider a general three-dimensional system whose state is characterized by the function $\psi(x, y, z, t)$. The dynamics of this system is governed by the extremum principle

$$\begin{aligned} I[\psi] &\equiv \int_{t_0}^{t_1}\iiint_{\mathcal{V}} \mathcal{L}\left(t, x^i, \frac{\partial\psi}{\partial t}, \frac{\partial\psi}{\partial x^i}\right) dx^1\, dx^2\, dx^3\, dt \\ &= \text{extremum!} \end{aligned}$$

Here $\mathcal{L}$, the Lagrangian density of the system, may depend on time as well as the space coordinates $x^i, i = 1, 2, 3$. The integration is over the spatial volume $\mathcal{V}$ which is spanned by the coordinates x^1, x^2, and x^3.

Nature dictates that the actual performance (i.e. the evolution in time) of the system is that one which is optimal in the sense that it extremizes the variational integral $I[\psi]$.

This means that if $\psi(t, x^i)$ is such an optimal evolution, then the variational integral will be unchanged to first order if one evaluates this integral for a variant evolution, say $\psi(t, x^i) + \delta\psi(t, x^i)$. In other words

$$I[\psi + \delta\psi] - I[\psi] \equiv \delta I[\psi] = O[\delta\psi]$$

where $O[\delta\psi]$ is a quantity of second or higher order in $\delta\psi$. It is, however, understood that the optimal evolution and its variant agree initially at t_0 and finally at t_i

$$\begin{aligned} \psi(t_0, x^i) &= \psi(t_0, x^i) + \delta\psi(t_0, x^i) \\ \psi(t_1, x^i) &= \psi(t_1, x^i) + \delta\psi(t_1, x^i). \end{aligned}$$

In other words the variation $\delta\psi(t, x^i)$ vanishes at the ends of the time interval,

$$\left.\begin{aligned} \delta\psi(t_0, x^i) &= 0 \\ \delta\psi(t_1, x^i) &= 0 \end{aligned}\right\} \quad \forall\ (x^1, x^2, x^3) \in \mathcal{V}. \tag{3.40}$$

Furthermore it is also understood that the variants $\psi + \delta\psi = \psi$ on the boundary for all times $t_0 \leq t \leq t_1$. In other words, the variation $\delta\psi(t, x^i)$ satisfies the homogeneous boundary condition that for all times

$$\delta\psi(t, x^i) = 0 \quad \text{whenever} \quad (x^1, x^2, x^3) \in \partial\mathcal{V}. \tag{3.41}$$

Here $\partial\mathcal{V} \equiv$ boundary of $\mathcal{V}$.

Equations (3.40) and (3.41) can be restated by observing that $\delta\psi(t, x^i)$ is in general non-zero in the interior of the four-dimensional spacetime region $[t_0, t_1] \times \mathcal{V}$, but

$$\boxed{\delta\psi(t, x^i) = 0 \quad (t, x^1, x^2, x^3) \in \partial\big([t_0, t_1] \times \mathcal{V}\big)}$$

i.e. $\delta\psi = 0$ whenever (t, x^1, x^2, x^3) lies on the *boundary* of the four-dimensional spacetime cube $[t_0, t_1] \times \mathcal{V}$.

[FIGURE]

3.13.2 Euler-Lagrange Equation

Subject to these *boundary conditions* on the variants, the optimal evolution satisfies then the necessary condition

$$\delta I[\psi] = 0.$$

We now resort in our customary fashion to the familiar calculus of variations technique to obtain the Euler-Lagrange equation implied by this necessary condition.

The variation in the Lagrangian density due to the variation $\delta\psi$ is

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\dot{\psi}}\delta\big(\tfrac{\partial\psi}{\partial t}\big) + \sum_{i=1}^{3}\frac{\partial\mathcal{L}}{\partial\big(\frac{\partial\psi}{\partial x^i}\big)}\delta\big(\tfrac{\partial\psi}{\partial x^i}\big).$$

Hamilton's principle therefore becomes

$$0 = \delta I = \int_1^2\iiint_{\mathcal{V}}\Big\{\frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \underbrace{\frac{\partial\mathcal{L}}{\partial\dot{\psi}}\delta\big(\tfrac{\partial\psi}{\partial t}\big)}_{\boxed{1}} + \sum_{i=1}^{3}\underbrace{\frac{\partial\mathcal{L}}{\partial\big(\frac{\partial\psi}{\partial x^i}\big)}\delta\big(\tfrac{\partial\psi}{\partial x^i}\big)}_{\boxed{2}}\Big\}d^3x\,dt.$$

Make use of the same integration by parts occurring in the derivation of the ordinary Euler-Lagrange equations:

$$\boxed{1} = \frac{\partial\mathcal{L}}{\partial\dot{\psi}}\underbrace{\left[\frac{\partial(\psi+\delta\psi)}{\partial t} - \frac{\partial\psi}{\partial t}\right]}_{\frac{\partial}{\partial t}(\delta\psi)} = -\int\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}}\right)\delta\psi\,dt.$$

The integral involving the spatial derivatives can be manipulated in a similar manner:

$$\boxed{2} = \int \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x^k}\right)} \left[\frac{\partial(\psi+\delta\psi)}{\partial x^k} - \frac{\partial\psi}{\partial x^k}\right] dx^k = \int \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x^k}\right)} \frac{\partial}{\partial x^k}(\delta\psi)\, dx^k$$

An integration by parts yields:

$$\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x^k}\right)}\delta\psi\bigg|_{\partial\mathcal{R}} - \int \frac{\partial}{\partial x^k}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x^k}\right)}\delta\psi\right] dx^k. \qquad \underset{\text{no sum}}{k=1,2,3}$$

The integrated term vanishes because it is evaluated at the two end points of the x^k interval, and they lie on the boundary $\partial\mathcal{R}$.

The variation in I becomes therefore

$$\delta I = \int_1^2 \iiint_{\mathcal{R}} \left\{\frac{\partial \mathcal{L}}{\partial\psi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\dot{\psi}}\right) - \sum_{k=1}^{3} \frac{\partial}{\partial x^k}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x^k}\right)}\right\} \delta\psi\ dt\, d^3x.$$

The optimal nature of ψ demands that $\delta I = 0$ for arbitrary ψ, in particular those variations $\delta\psi$ which have a "blip" function behaviour around $(t, x^i) \in [t_0, t_1] \times \mathcal{V}$. Consequently the variational derivative of I,

$$\boxed{0 = \frac{\delta I}{\delta\psi}\delta I = \frac{\partial \mathcal{L}}{\partial\psi} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\dot{\psi}} - \sum_{k=1}^{3} \frac{\partial}{\partial x^k}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x^k}\right)}},$$

must vanish.

Note: If we let

$$\left.\begin{array}{ll} x^0 & \equiv t \\ x^1 & = x^1 \\ x^2 & = x^2 \\ x^3 & = x^3 \end{array}\right\} x^\mu \qquad \mu = 0,1,2,3$$

then the Euler-Lagrange equation can be written in the more symmetric form

$$\boxed{\frac{\partial \mathcal{L}}{\partial\psi} - \sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x^\mu}\right)} = 0}.$$

Comment.
(i) These Euler-Lagrange equations have a structure that puts the time on an equal status with the space coordinates.

Furthermore, the action integral

$$I[\psi] = \iiint \mathcal{L}\, d^4x$$

[FIGURE]

has been extremized on the set of all functions $\psi(x^\mu)$ which have fixed values on the 3-dimensional boundary of a given 4-dimensional region. It turns out that the general shape can be, within certain limits, fairly general.
(ii) If the action is a functional of several functions, so that

$$\mathcal{L} = \mathcal{L}\Big(x^\mu, \psi^1, \cdots, \psi^s, \frac{\partial \psi^1}{\partial x^\mu}, \cdots, \frac{\partial \psi^s}{\partial x^\mu}\Big)$$

then the Euler equations for the functional are

$$\boxed{\frac{\partial \mathcal{L}}{\partial \psi^n} - \sum_{\mu=0}^{3} \frac{\partial}{\partial \mu^k} \frac{\partial \mathcal{L}}{\partial\big(\frac{\partial \psi^n}{\partial \mu^k}\big)} = 0} \qquad n = 1, \ldots, s.$$

3.13.3 Examples and Applications

Example 1. *The Wave Equation.*

Consider the action for the wave equation,

$$I = \frac{1}{2} \iiiint \left[\left(\frac{\partial \psi}{\partial t}\right)^2 - \left(\frac{\partial \psi}{\partial x}\right)^2 - \left(\frac{\partial \psi}{\partial y}\right)^2 - \left(\frac{\partial \psi}{\partial z}\right)^2 \right] dt\, dx\, dy\, dz.$$

The E-L equation yields

$$\boxed{-\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0},$$

the familiar wave equation.

Example 2. *The (Time Independent) Helmholtz equation.*

Consider the variational problem

$$\iiint \mathcal{F} dx\, dy\, dz = \iiint \left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 \right] dx\, dy\, dz = \text{extremum}$$

for all those functions ψ which are constrained by the condition

$$\iiint \psi^2 dx\, dy\, dz = 1$$

This problem is solved by the method of Lagrange multipliers. Let the variational density be

$$\mathcal{F}' = \mathcal{F} - \lambda\psi^2.$$

The Euler equation for this integrand is

$$\frac{\partial \mathcal{F}'}{\partial \psi} - \sum_{i=1}^{3} \frac{d}{dx^i} \frac{\partial \mathcal{F}'}{\partial\left(\frac{\partial \psi}{\partial x^i}\right)} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + \lambda\psi = \nabla^2\psi + \lambda\psi = 0,$$

which is the Helmholtz equation.

Example 2. *Helmholtz equation relative to arbitrary coordinates.*

Suppose we wish to rewrite this equation in terms of the new curvilinear coordinates

$$\begin{aligned} \bar{x}^1 &= \bar{x}^1(x^1, x^2, x^3) \\ \bar{x}^2 &= \bar{x}^2(x^1, x^2, x^3) \\ \bar{x}^3 &= \bar{x}^3(x^1, x^2, x^3) \end{aligned}$$

(a) Instead of applying this transformation to the differential equation, apply it to the variational principle, and then write down the Euler equation relative to these new coordinates:

$$\begin{aligned} \mathcal{F} = \nabla\psi \cdot \nabla\psi &= \left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} \Bigg(\underbrace{\frac{\partial \bar{x}^i}{\partial x^1}\frac{\partial \bar{x}^j}{\partial x^1} + \frac{\partial \bar{x}^i}{\partial x^2}\frac{\partial \bar{x}^j}{\partial x^2} + \frac{\partial \bar{x}^i}{\partial x^3}\frac{\partial \bar{x}^j}{\partial x^3}}_{\equiv g^{ij} \text{ where } g^{ij}g_{jk} = \delta^i_k} \Bigg) \frac{\partial \psi}{\partial \bar{x}^i}\frac{\partial \psi}{\partial \bar{x}^j}. \end{aligned}$$

It is easy to show that

$$\left[\frac{\partial \bar{x}^i}{\partial x^1}\frac{\partial \bar{x}^j}{\partial x^1} + \frac{\partial \bar{x}^i}{\partial x^2}\frac{\partial \bar{x}^j}{\partial x^2} + \frac{\partial \bar{x}^i}{\partial x^3}\frac{\partial \bar{x}^j}{\partial x^3}\right] \equiv \left[g^{ij}\right]$$

is the matrix inverse to $[g_{ij}]$, which defined the metric

$$\begin{aligned} \left(ds\right)^2 &= \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 \\ &= \delta_{ij}\, dx^i\, dx^j \\ &= \delta_{ij} \frac{\partial x^i}{\partial \bar{x}^k}\frac{\partial x^j}{\partial \bar{x}^\ell}\, d\bar{x}^k\, d\bar{x}^\ell \\ &= g_{k\ell}\, d\bar{x}^k\, d\bar{x}^\ell. \end{aligned}$$

In other words,

$$\boxed{\left[g^{ij}\right]\left[g_{jk}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

Thus relative to these new coordinates $\bar{x}^i$ one has

$$\boxed{\nabla\psi \cdot \nabla\psi = g^{ij}\frac{\partial\psi}{\partial\bar{x}^i}\frac{\partial\psi}{\partial\bar{x}^j}}$$

which is easy to write down once g_{ij} is known.

[FIGURE]

(b) The volume element relative to the new coordinates is expressed in terms of the Jacobian determinant J:

$$\begin{vmatrix} \frac{\partial x^1}{\partial\bar{x}^1}\,d\bar{x}^1 & \frac{\partial x^2}{\partial\bar{x}^1}\,d\bar{x}^1 & \frac{\partial x^3}{\partial\bar{x}^1}\,d\bar{x}^1 \\ \frac{\partial x^1}{\partial\bar{x}^2}\,d\bar{x}^2 & \frac{\partial x^2}{\partial\bar{x}^2}\,d\bar{x}^2 & \frac{\partial x^3}{\partial\bar{x}^2}\,d\bar{x}^2 \\ \frac{\partial x^1}{\partial\bar{x}^3}\,d\bar{x}^3 & \frac{\partial x^2}{\partial\bar{x}^3}\,d\bar{x}^3 & \frac{\partial x^3}{\partial\bar{x}^3}\,d\bar{x}^3 \end{vmatrix} = \det J\; d\bar{x}^1\,d\bar{x}^2\,d\bar{x}^3, \quad \text{where} \quad J = \left[\frac{\partial x^i}{\partial\bar{x}^j}\right].$$

This determinant is readily available from the transformed metric coefficients:

$$g_{k\ell} = \frac{\partial x^i}{\partial\bar{x}^k}\;\delta_{ij}\;\frac{\partial x^k}{\partial\bar{x}^\ell}$$

or

$$G = J^T \quad I \quad J$$

so that

$$\det G = (\det J)^2$$

or

$$\det J = \sqrt{\det G} \equiv \sqrt{g}\ .$$

(c) The transformed variational integral is

$$\iiint g^{k\ell}\frac{\partial\psi}{\partial\bar{x}^k}\frac{\partial\psi}{\partial\bar{x}^\ell}\sqrt{g}\;d\bar{x}^1\,d\bar{x}^2\,d\bar{x}^3 = \text{extremum}$$

and the normalization condition is

$$\iiint \psi^2\sqrt{g}\;d\bar{x}^1\,d\bar{x}^2\,d\bar{x}^3 = 1.$$

(d) The Euler-Lagrange equations are

$$\frac{\partial}{\partial x^k}\left[\sqrt{g}\,g^{k\ell}\frac{\partial\psi}{\partial x^\ell}\right] + \lambda\sqrt{g}\,\psi = 0$$

or

$$\boxed{\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left[\sqrt{g}\, g^{k\ell}\frac{\partial \psi}{\partial x^\ell}\right] + \lambda\psi = 0}.$$

Comparing this equation with the one in example 2, we see that

$$\boxed{\nabla^2\psi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left[\sqrt{g}\, g^{k\ell}\frac{\partial \psi}{\partial x^\ell}\right]}$$

is the expression for the Laplacian relative to curvilinear coordinates.

Example: *Find the Laplacian relative to spherical coordinates.*

Step 1: Write down the metric relative to these coordinates.

$$(ds)^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

$$\sqrt{g} = \sqrt{\det g_{ij}} = \sqrt{r^4 \sin^2\theta} = r^2 \sin\theta$$

Step 2:

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{bmatrix}.$$

Thus

$$\nabla^2\psi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r} r^2 \sin\theta \frac{\partial\psi}{\partial r} + \frac{\partial}{\partial\theta}\frac{r^2\sin\theta}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{\partial}{\partial\phi}\frac{r^2\sin\theta}{r^2\sin^2\theta}\frac{\partial\psi}{\partial\phi}\right]$$

$$\boxed{\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial\psi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}}.$$

Lecture 16

Chapter 4

DIRECT METHODS IN THE CALCULUS OF VARIATIONS

We know how to minimize the functional $J[\psi] = \iiint \mathcal{L}(x^i, \psi, \frac{\partial \psi}{\partial x^i})\, d^3x$ indirectly. We write down the Euler equation, a differential equation for $\psi(x^i)$. Then we solve that equation by finding that function $\psi(x^i)$ which satisfies the conditions at the boundary. This can be a daunting task because what one needs to do is solve the differential equation in the *whole* region enclosed by the boundary, even if one wishes to find only an approximate solution.

The ultimate object of interest is a function which *extremizes* $J[\psi]$ and *not* necessarily a function which *satisfies the Euler equation*. Consequently, it is appropriate to ask:

> "Can one not find this optimizing function (if not exactly, then at least approximately) without having to bother with the intermediate expense of having to integrate the Euler equation over a finite multidimensional region?"

The answer to this question leads us from the *indirect variational* methods based on solving partial differential equations to *direct variational* methods which dispense with these equations altogether.

The direct methods have two additional attractive features:

(i) they can be used to solve differential equations and

(ii) they can be used to calculate a solution to the differential equation with arbitrary accuracy.

Thus suppose that one can show by some means that the functional $J[\psi]$ has an extremum for a sufficiently smooth admissible function ψ. Then this very fact proves the existence of a solution to the corresponding Euler differential equation, and thereby gives us a rough idea about the nature of the solution.

4.1 Minimizing Sequence.

The general idea behind the direct method is as follows.

Consider the problem of finding the minimum of a functional $J[y]$ on the space $\mathcal{M}$ of allowed ("admissible") functions. (This usually means the functions must be sufficiently differentiable.) We assume (a) for each of these functions the functional J is *finite*, i.e.

$$J[y] < \infty$$

and (b) $J[y]$ has a *minimum*, i.e. $\exists \mu$ such that

$$\inf_{y \in \mathcal{M}} J[y] = \mu > -\infty.$$

Here $\inf_{y \in \mathcal{M}} J[y]$ is the *greatest lower bound* ("infimum") for the set $\{J[y] : y \in \mathcal{M}\}$; in other words

$$\inf_{y \in \mathcal{M}} J[y] = \max\{m : J[y] \geq m; y \in \mathcal{M}\}.$$

[FIGURE]

The existence of such a greatest lower bound implies that $\exists$ a sequence of functions in $\mathcal{M}$ (a "minimizing sequence")

$$y_1, y_2, \ldots$$

such that

$$\lim_{n \to \infty} J[y_n] = \mu.$$

[FIGURE]

If the sequence $y_1, y_2,$ *dots* has a limit, say $\hat{y}$:

$$\lim_{n \to \infty} y_n = \hat{y},$$

then one can write

$$\lim_{n \to \infty} J[y_n] = J[\hat{y}]$$

$$\lim_{n \to \infty} J[y_n] = J\Big[\lim_{n \to \infty} y_n\Big]$$

or $J[\hat{y}] = \mu$.

In that case $\hat{y}$ is the solution to the problem and the elements of the minimizing sequence y_n can be considered as approximations to the problem.

[FIGURE]

4.2 Implementation via Finite-Dimensional Approximation

An implementation of this idea is as follows: Consider the following sequence

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M}_n \subset \cdots \subset \mathcal{M}$$

of spaces of functions

$$\begin{array}{rcll} \mathcal{M}_1 & = & \{ & y_1 = y_1(x^i, \alpha_1) \quad -\infty < \alpha_1 < \infty \ \} \\ \mathcal{M}_2 & = & \{ & y_2 = y_2(x^i, \alpha_1, \alpha_2) \quad -\infty < \alpha_1, \alpha_2 < \infty \\ & & & y_2(x^i, \alpha_1, 0) = y_1(x^i, \alpha_1) \ \} \\ \mathcal{M}_3 & = & \{ & y_3 = y_3(x^i, \alpha_1, \alpha_2, \alpha_3) \quad -\infty < \alpha_1, \alpha_2, \alpha_3 < \infty \\ & & & y_3(x^i, \alpha_1, \alpha_2, 0) = y_1(x^i, \alpha_1, \alpha_2) \ \} \\ & \vdots & & \end{array}$$

Thus each $\mathcal{M}_n$ is coordinatized by its n coordinates $(\alpha_1, \cdots, \alpha_n)$.

The functional J can be minimized on each subspace $\mathcal{M}_n$. Thus

(1) on the 1-dimensional space $\mathcal{M}_1$ consider $\min_{y \in \mathcal{M}_1} J[y] = \mu_1$. This is obtained by letting

$$J(\alpha_1) = J[y_1(\alpha_1)]$$

requiring

$$\frac{\partial J(\alpha_1)}{\partial \alpha_1} = 0,$$

and then letting $\mu_1 = J(\alpha_1)$.

(2) on the 2-dimensional space $\mathcal{M}_2$ consider $\min_{y \in \mathcal{M}_2} J[y] = \mu_2$. This is obtained by letting

$$J(\alpha_1, \alpha_2) = J[y_2(\alpha_1, \alpha_2)]$$

requiring

$$\frac{\partial J(\alpha_1, \alpha_2)}{\partial \alpha_i} = 0 \qquad i = 1, 2,$$

and then letting $\mu_2 = J(\alpha_1, \alpha_2)$.

$\vdots$

(n) on the n-dimensional space $\mathcal{M}_n$ consider $\min_{y\in\mathcal{M}_n} J[y] = \mu_n$. This is obtained by letting

$$J(\alpha_1, \ldots, \alpha_n) = J[y_n(\alpha_1, \ldots, \alpha_n)]$$

requiring

$$\frac{\partial J(\alpha_1, \ldots, \alpha_n)}{\partial \alpha_i} = 0 \qquad i = 1, \ldots, n,$$

and then letting $\mu_n = J(\alpha_1, \ldots, \alpha_n)$.

$\vdots$

This results in a non-increasing sequence of minima

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq \cdots \geq \inf_{y\in\mathcal{M}} J[y] = \hat{\mu}$$

because

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots \subset \mathcal{M}_n \subset \cdots \subset \mathcal{M}.$$

In other words, the minimum can never increase as we increase the size of the function space.

Lecture 17

4.3 Raleigh's Variational Principle

Let us apply the direct method to one of the most important variational problems, namely the one governing a linear vibrating system, say, a vibrating compressible medium enclosed in a three-dimensional volume:

$$\iiint (\nabla\psi)\cdot(\nabla\psi)\, dV = \text{extremum}$$

subject to the constraint

$$\iiint \psi^2\, dV = 1; \quad \psi = 0 \quad \text{on the boundary of the volume.}$$

This variational problem extremizes the potential energy of the system at a moment of time symmetry (see page 58 in Section 2.3.1). The Euler equation

$$\nabla^2\psi + \lambda\psi = 0 \tag{4.1a}$$

with the boundary condition

$$\psi = 0 \quad \text{on the boundary of the volume,} \tag{4.1b}$$

is what in one dimension corresponds to the Sturm-Liouville boundary value problem. Here we have its three-dimensional version.

The interesting feature about this variational problem is that it has many (discrete) solutions. In other words, we are asked to find not only a function which *minimizes* the variational integral, but also those functions which correspond to *saddle points* in the function space on which the integral is defined.

4.3.1 The Raleigh Quotient

In order to help us see the relationship between these different solution and help us develop methods for determining them at least approximately, we shall use a powerful device which is called the Raleigh Quotient after the person who invented it and made extensive use of it in analyzing vibrational problems.

The Raleigh Quotient for our variational problem consists of

$$\frac{\iiint (\vec{\nabla}\psi)\cdot(\vec{\nabla}\psi)\,dV}{\iiint \psi^2\,dV} \equiv \frac{N}{D}.$$

The utility of this quotient lie in the fact that the variational problem ("Raleigh's variational principle")

$$\frac{N}{D} = \text{extremum}$$

subject to

$$\psi = 0 \quad \text{on the boundary,}$$

is equivalent to the original variational problem, Eqs. (4.1). This is a big advantage because Raleigh's variational problem permits us to consider variants of ψ which are arbitrary instead of being restricted by the constraint condition

$$\iiint \psi^2\,dV = 1.$$

The *first* significant property of the Raleigh quotient lies in the fact that if ψ satisfies the eigenvalue ("Helmholtz") equation

$$\nabla^2\psi = -k^2\psi,$$

then

$$\boxed{\frac{N}{D} = k^2}.$$

Indeed, using Gauß's divergence theorem and the boundary condition that $\psi = 0$ on the boundary we find

$$\begin{aligned}
\frac{N}{D} &= \frac{\iiint (\nabla\psi)\cdot(\nabla\psi)\,dV}{\iiint \psi^2\,dV} \\
&= \frac{\iiint [\nabla\cdot(\psi\vec{\nabla}\psi) - \psi\nabla\cdot\vec{\nabla}\psi]\,dV}{\iiint \psi^2\,dV} \\
&= \frac{-\iiint \psi\nabla^2\psi\,dV + \iint \psi\vec{\nabla}\psi\cdot\vec{n}\,dA}{\iiint \psi^2\,dV} \\
&= \frac{-\iiint \psi(-k^2\psi)\,dV + 0}{\iiint \psi^2\,dV} \\
&= k^2.
\end{aligned}$$

Remark. If ψ satisfies the homogeneous boundary condition

$$\vec{n}\cdot\vec{\nabla}\psi + f\psi = 0,$$

then the Raleigh quotient is

$$\frac{N}{D} = \frac{\iiint (\nabla\psi)^2\,dV + \iint f\psi^2\,dA}{\iiint \psi^2\,dV} \tag{4.2}$$

instead. But we still have the basic fact,

$$\left.\begin{aligned} \nabla^2\psi + k^2\psi &= 0 \\ \vec{n}\cdot\vec{\nabla}\psi + \psi\big|_{\text{boundary}} &= 0 \end{aligned}\right\} \quad \text{implies} \quad \frac{N}{D} = k^2.$$

The *second* significant property of the Raleigh quotient is that extremizing it is equivalent to the original extremum principle; in other words, if ψ is an admissible function (i.e. it satisfies, among others, the boundary conditions) which extremizes the Raleigh quotient, then this function ψ satisfies the Helmholtz equation. More briefly, we have

$$\frac{N}{D} = \text{extremum} \qquad \psi\big|_{\text{boundary}}$$

implies that

$$\nabla^2\psi + k^2\psi = 0$$

where

$$k^2 = \frac{N}{D}.$$

The variational calculation leading to this conclusion is based on the Raleigh variational principle

$$\delta\left(\frac{N}{D}\right) = 0. \qquad \text{("Raleigh variational principle")}$$

The details are easy to perform:

$$\delta\left(\frac{N}{D}\right) = \frac{\delta N}{D} - \frac{N}{D}\frac{\delta D}{D}.$$

Using the fact that $\delta\psi = 0$ on the boundary, one has

$$\begin{aligned}
\delta N &= \delta \iiint \nabla\psi \cdot \nabla\psi \, dV \\
&= 2 \iiint \nabla\psi \cdot \nabla\delta\psi \, dV \\
&= 2 \iiint \left[\nabla \cdot (\delta\psi \, \vec{\nabla}\psi) - \nabla \cdot \vec{\nabla}\psi \; \delta\psi\right] dV \\
&= -2 \iiint (\nabla^2\psi) \, \delta\psi \, dV + 2 \iint \vec{n} \cdot \nabla\psi \; \delta\psi \, dA \\
&= -2 \iiint (\nabla^2\psi) \, \delta\psi \, dV + 0
\end{aligned}$$

$$\delta D = 2 \iiint \psi \, \delta\psi \, dV$$

Thus

$$0 = \delta\left(\frac{N}{D}\right) = -\frac{2}{D} \iiint \left[\nabla^2\psi + \frac{N}{D}\psi\right] \delta\psi \, dV.$$

Inside the integration volume, the variation $\delta\psi$ is arbitrary. Consequently,

$$\nabla^2\psi + \frac{N}{D}\psi = 0$$

or

$$\nabla^2\psi + k^2\psi = 0$$

if we set

$$k^2 = \frac{N}{D}.$$

Remark. If the trial functions ψ are required to satisfy the homogeneous boundary condition

$$\vec{n} \cdot \nabla\psi + f\psi\big|_{\text{boundary}} = 0$$

on the boundary, then Raleigh's variational principle still holds provided one uses the corresponding Raleigh quotient, Eq. (4.2).

Remark. $\delta\left(\frac{N}{D}\right) = 0$ implies the original variational principle, Eqs. (4.1): $\delta(N - k^2 D) = 0$. The *third* significant property of the Raleigh quotient is that the variational principle based on it reveals the relationship between the various solutions which extremize it.

1. We note that when
$$k^2 = \frac{\iiint (\nabla\psi)^2 \, dV}{\iiint \psi^2 \, dV}$$
is evaluated for all possible trial functions ψ, there will exist an *absolute minimum*, k_0^2. Let ψ_0 be that optimal trial function corresponding to this minimum:
$$\psi_0 \longleftrightarrow k_0^2$$

2. It is not difficult to show that: If k_i^2 and k_j^2 are two eigenvalues of $(\nabla^2 + k^2)\psi = 0$, then
$$k_i^2 + k_j^2 \Rightarrow \iiint \psi_i \, \psi_j \, dV = 0$$

2b. Thus one may obtain a variational principle for k_1^2, where $k_1^2 > k_0^2$ but $k_n^2 > k_1^2 \quad (n > 1)$. One simply imposes the additional constraint
$$\iiint \psi \, \psi_0 \, dV = 0$$
on the set of trial functions. Then
$$\frac{\iiint (\nabla\psi)^2 \, dV}{\iiint \psi^2 \, dV} = \text{minimum}$$
subject to
$$\iiint \psi \, \psi_0 \, dV = 0$$
yields
$$k_1^2 \qquad \text{and} \qquad \psi_1.$$

This process may be continued to obtain k_2^2 and ψ_2, and other eigenfunctions and eigenvalues:
$$k_0^2 \leq k_1^2 \leq k_2^2 \leq \cdots$$
$$\psi_0, \quad \psi_1, \quad \psi_2, \quad \cdots$$

It is therefore evident that the Raleigh quotient reveals:

(a) the *existence* of an *infinite* sequence of eigenvalues and eigenfunctions of the boundary value problem

$$\nabla^2\psi + k^2\psi = 0$$

(b) the sequence of eigenvalues is *non-decreasing*

(c) the sequence of functions which extremize the variational integral are *orthogonal*.

4.3.2 Raleigh-Ritz Principle

The Raleigh variational principle based on the Raleigh quotient furnishes us with a framework for an *approximate determination* of the eigenvalues and their eigenfunctions. This framework is a variational method which consists of letting the trial function ("variant")

$$\phi = \phi(x, \alpha_1, \ldots, \alpha_s)$$

depend on a finite number of parameters $\alpha_1, \ldots, \alpha_s$. The dependence of ϕ on α_i may be a non-linear one or a linear one.

The Raleigh-Ritz Method is a special method. It consists of letting the trial functions depend *linearly* on these parameters,

$$\phi = \sum_{i=1}^{s} \alpha_i \phi_i(x).$$

Let us introduce the following matrix elements:

$$\int \phi_i \, \phi_j \, dV \equiv B_{ij}$$

and

$$\int \vec{\nabla}\phi_i \cdot \vec{\nabla}\phi_j \, dV = -\int \phi_i \nabla^2 \phi_j \, dV \equiv A_{ij}.$$

We now take advantage of the fact that Raleigh's variational principle

$$\delta\left(\frac{N}{D}\right) = 0$$

implies

$$\delta(N - k^2 D) = 0.$$

This is the condition for extremizing the expression

$$N - k^2 D = \sum_{i=1}^{s}\sum_{j=1}^{s}(A_{ij} - k^2 B_{ij})\alpha_i\alpha_j \equiv J(\alpha_1, \ldots, \alpha_s).$$

Thus we have

$$\frac{\partial J}{\partial x^i} = 0 \qquad i = 1, \ldots, s.$$

This results in

$$2\sum_{j=1}^{s}(A_{ij} - k^2 B_{ij})\alpha_j = 0 \qquad i = 1, \ldots, s$$

or

$$[A]\,\vec{\alpha} = k^2\,[B]\,\vec{\alpha}.$$

These equations have a solution if

$$\det\left|A_{ij} - k^2 B_{ij}\right| = 0.$$

This is the familiar secular equation for the symmetric matrices $[A_{ij}]$ and $[B_{ij}]$. The solutions are eigenvalues which approximate those determined by the problem

$$(\nabla^2 + k^2)\psi = 0,$$

plus homogeneous Dirichlet or Neumann boundary conditions.

The solution consists of an approximation to the first s eigenvalues:

$$k_0^2 \le k_1^2 \le \cdots \le k_s^2$$

and their corresponding approximate solutions

$$\phi^{(0)} = \sum \alpha_i^{(0)}\phi_i(x), \quad \ldots, \quad \phi^{(s))} = \sum \alpha_i^{(s)}\phi_i(x)$$

where $\alpha_i^{(m)}$ is an eigenvalue for the problem

$$[A]\,\vec{\alpha}^{(m)} = (k_m)^2\,[B]\,\vec{\alpha}^{(m)}.$$

4.3.3 Vibration of a Circular Membrane

Consider a clamped circular membrane of radius a. The variational principle for this

[FIGURE]

problem is

$$[k^2] = \frac{\iiint(\nabla\psi)^2\,dV}{\iiint\psi^2\,dV} \equiv \frac{N}{D}.$$

The amplitude ψ satisfies

$$\psi(a) = 0$$

with ψ and $\nabla\psi$ *continuous* inside the disc. For algebraic simplicity, consider *only circularly symmetric* modes. Thus

$$\psi = \psi(r)$$

is a function of r only, independent of angle.

The variational principle becomes

$$k^2 = \frac{\int_0^a (\frac{\partial\psi}{\partial r})^2 \, r \, dr}{\int_0^a \psi^2 \, r \, dr} \equiv \frac{N}{D}.$$

Comment: The differential equation corresponding to this problem is

$$\frac{1}{r}\frac{d}{dr} r \frac{d\psi}{dr} + k^2\psi = 0$$

whose eigenvalues k^2 are determined by $J_0(ka) = 0$. It is advantageous to introduce the dimensionless independent variable

$$x \equiv \frac{r}{a}.$$

Consequently

$$(ka)^2 = \frac{\int_0^1 (\frac{\partial\psi}{\partial x})^2 \, x \, dx}{\int_0^1 \psi^2 \, x \, dx} \equiv \frac{N}{D}.$$

(A) *Lowest* eigenvalue and eigenfunction: Consider the possible trial functions. The simplest one which vanishes at $x = 1$ and which has *continuous gradient at* $x = 0$ is

$$\phi_1 = 1 - x^2.$$

Comment: $1 - x$ won't do because its slope at $x = 0$ is -1, giving rise to a cusp at the center.

Insert ϕ_1 and obtain

$$\frac{1}{r}\frac{d}{dr} r \frac{d\psi}{dr} + k^2\psi = 0$$

$$(ka)^2 = \frac{\int_0^1 (-2x)^2 \, x \, dx}{\int_0^1 (1-x^2)^2 \, x \, dx} = \frac{\int_0^1 4x^3 \, dx}{-\frac{(1-x^2)^2}{6}\Big|_0^1} = \frac{x^4\big|_0^1}{\frac{1}{6}} = 6.$$

The *exact* value of $(ka)^2$ is determined by the first zero of J_0:

$$J_0(ka) = 0 \Rightarrow \quad \overset{\text{Lowest root}}{k_0 a = 2.40483;} \quad \overset{\text{Next lowest root}}{k_1 a = 5.52008}$$
$$(k_0 a)^2 = 5.78319; \quad (k_1 a)^2 = 30.4713$$

Thus the error in the crude variational estimate for the lowest eigenvalue is

$$\frac{6-5.78319}{6}=\frac{0.21681}{6}\approx 0.036, \qquad \text{i.e.} \qquad 3.6\%$$

(B) The next approximation uses an improved trial function

$$\phi=\alpha_1\left(1-x^2\right)+\alpha_2\left(1-x^2\right)^2.$$

This choice is based on the fact that any function which (a) is zero at $x=1$ and (b) has zero slope at $x=0$ can be expanded in a *power series in* $(1-x^2)$. *It is observations like these about the (geometrical) nature of trial functions which makes the variational principle practical.*

Insert this trial function into the expression for $(ka)^2$ and obtain

$$\frac{N}{D}=(ka)^2\equiv\lambda=\frac{\vec{\alpha}^{\,T}A\,\vec{\alpha}}{\vec{\alpha}^{\,T}B\,\vec{\alpha}}$$

or

$$A_{11}\alpha_1^2+2A_{12}\alpha_1\alpha_2+A_{22}\alpha_2^2=\lambda\left(B_{11}\alpha_1^2+2B_{12}\alpha_1\alpha_2+B_{22}\alpha_2^2\right).$$

Here

$$\begin{aligned}
B_{11}&=\int_0^1\left(1-x^2\right)^2 x\,dx=\frac{1}{6}\\
B_{12}&=\int_0^1\left(1-x^2\right)^3 x\,dx=\frac{1}{8}\\
B_{22}&=\int_0^1\left(1-x^2\right)^4 x\,dx=\frac{1}{10}\\
A_{11}&=\int_0^1\left[\frac{d}{dx}(1-x^2)\right]^2 x\,dx=\frac{1}{6}\\
A_{12}&=\int_0^1\frac{d}{dx}\left(1-x^2\right)\frac{d}{dx}\left(1-x^2\right)^2 x\,dx=\frac{2}{3}\\
A_{22}&=\int_0^1\left[\frac{d}{dx}\left(1-x^2\right)^2\right]^2 x\,dx=\frac{2}{3}.
\end{aligned}$$

Differentiate with respect to α_1 and α_2 and set

$$\begin{aligned}
\frac{\partial}{\partial\alpha_1}\left(\frac{N}{D}\right)&=\frac{\partial(ka)^2}{\partial\alpha_1}=0\\
\frac{\partial}{\partial\alpha_2}\left(\frac{N}{D}\right)&=\frac{\partial(ka)^2}{\partial\alpha_2}=0
\end{aligned}$$

to obtain

$$(A_{11} - \lambda B_{11})\alpha_1 + (A_{12} - \lambda B_{12})\alpha_2 = 0$$
$$(A_{21} - \lambda B_{21})\alpha_1 + (A_{22} - \lambda B_{22})\alpha_2 = 0$$

This is a pair of linear equations for α_1 and α_2. It has a solution only if $\det|\lambda B - A| = 0$, i.e. if

$$\det|\lambda B - A| = 0 \Rightarrow \begin{vmatrix} \frac{1}{6}\lambda - 1 & \frac{1}{8}\lambda - \frac{2}{3} \\ \frac{1}{8}\lambda - \frac{2}{3} & \frac{1}{10}\lambda - \frac{2}{3} \end{vmatrix} = 0.$$

This is a quadratic equation for $\lambda = (ka)^2$. There are two roots. The *smaller* root is

$$\lambda = (ka)^2 = 5.78413$$

which is very close to the exact value

$$\lambda_{\text{exact}} = (ka)^2_{\text{exact}} = 5.78319.$$

The ratio $\frac{\alpha_2}{\alpha_1}$ is, from the second equation,

$$\frac{\alpha_2}{\alpha_1} = -\frac{\frac{2}{3} - \frac{1}{8}\lambda}{\frac{2}{3} - \frac{1}{10}\lambda} = 0.639.$$

Thus the wave function is

$$\psi_0 = (1 - x^2) + 0.639\left(1 - x^2\right)^2.$$

One should compare the first trial function

$$\phi_1 = 1 - x^2$$

and the second trial function

$$\psi_1 = \frac{1}{1.639}\left[\left(1 - x^2\right) + 0.639\left(1 - x^2\right)^2\right]$$

with the exact wave function

$$J_0(2.40483x). \leftarrow \text{exact}$$

[FIGURE]

Comment:

1. The larger second root of the secular quadratic equation is $\lambda_1 = (k_1 a)^2 = 36.883$, to be compared with the exact result $(k_1 a)^2_{\text{exact}} = 30.471$.

2. The approximate wave corresponding to this eigenvalue is orthogonal to the wave function determined by the first root.

$$\boxed{\lambda_1 = 36.883} \Rightarrow \frac{\alpha_2}{\alpha_1} = -\frac{\frac{2}{3} - \frac{\lambda}{8}}{\frac{2}{3} - \frac{\lambda}{10}}$$
$$= -\frac{\frac{2}{3} - 4.6103}{\frac{2}{3} - 3.6883} = -\frac{3.9437}{3.0216}$$
$$= -1.30516$$

The wave function is therefore

$$\psi_1 = (1 - x^2) - 1.30516\left(1 - x^2\right)^2.$$

This wave function is orthogonal to ψ_1:

$$\int_0^1 \psi_0\, \psi_1\, x\, dx = 0.$$

[FIGURE]