Math 665: Modern Mathematical Methods in Relativity Theory I

Tentative
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and Reading List

1. Rapid course in special relativity: Key ideas: Principle of relativity [T-W 3.1-3.3 (1.3)]; inertial (free float) frames [MTW Fig 1.7; T-W 2.2, 2.3, 2.4 (1.2, 1.4)]; Isotropy of space [T-W (1.3)].

2. Key ideas: Invariance of the interval [T-W 3.6, 3.7, 3.8 (1.5)].

3. Euclidean geometry vs. Lorentz geometry [T-W 5.2, 5.3 (1.1, 1.6)]; Lorentz transformations [T-W L.3, L.4, L.5, L.6]

4. Important mathematical properties of a Lorentz transformation. Causal classification of events [T-W 6.1, 6.2, 6.3 (1.7)].

5. Spacetime as a vector space; four velocity; four acceleration; wave propagation four vector. [Box 1.3, 2.4; Sect. 2.1-2.3 in MTW].

6. Significance and utility of the projection invariant; worldline of an accelerated observer [MTW Ch 6].

7. One parameter family; Instantaneous Lorentz frames along a given worldline. Fermi-Walker transport [MTW Ch. 6].

8. Fermi-Walker transport and curvilinear spacetime coordinates [MTW Ch. 6].

9. Covectors and Vectors: Definition of a linear function; the dual vector space; bracket notation; construction of linear functions; basis for dual space; no natural isomorphism between vector space and its dual; basis covectors = coordinate functions..[MTW 2.5 - 2.7, 9.1 - 9.5]

10. Bilinear functional; metric on a vector space; natural isomorphism between vectors and duals; basis representation of a metric; lowering the indices on the components of a vector; reciprocal vector basis; correspondence between reciprocal basis vectors and basis covectors; covariant vs. contravariant components of a vector.

11. Metric as an isomorphism between the vector space and its dual space; representation of the metric relative to a given basis; vector components related to those of its image in V^*; reciprocal basis. [MTW §2.4, Box 8.4, §13.2.]

12. Tensors as multilinear maps; components of a tensor relative to a chosen basis; tensor product; tensor product basis; space of tensors; constructing new tensors by "lowering" indeces, by contraction.

13. Examples of tensors of various rank; the Levi-Civita tensor; exterior product [MTW 3.5, 4.1-4.2]. Flux tensor in Euclidean space; Flux tube structure; particle (or charge) worldline density tensor. [Read MTW Box 4.2, Fig. 4.2-4.5, Box 4.4].

14. Particle (or charge) worldline density tensor; Symplectic inner product [MTW, Box 4.5]; Euclidean basis vs. symplectic basis [V. Arnold §41].

15. Euclidean rotation vs. Lorentz rotation vs. symplectic transformation [V. Arnold §42]; "Raising" and "lowering" of indices, contraction. Coordinate charts, atlas, manifold; coordinate representative of a function [MTW Ch. 9; Singer & Thorpe 97-99]; Example: Minkowski spacetime with a Rindler atlas.

16. Example of coordinate charts: Minkowski spacetime with Rindler charts, coordinate functions. Tangent vector [Singer & Thorpe §5.1; Hicks Ch. 1; MTW Ch. 9].

17. Tangent vector; transforming its components; vector as a derivation [Singer & Thorpe §5.1; Hicks Ch. 1].

18. Vector field, differential 1-form [MTW 2.5, 9.1] [MTW 2.5, 9.1-9.5]; tangent to a curve; integral curves; [Singer & Thorpe §5.1; Hicks Ch. 1].

19. The Rotation Group SO(3) as a manifold with three independent nowhere zero vector fields [MTW Problem 9.13].

20. Integral curve of a vector field [Singer & Thorpe pp. 125-126]. Commutator of two vector fields [MTW Box 8.4 E, Box 9.2; §9.6; Singer & Thorpe p. 127]. The differential (one form) of a function [MTW §9.4; T. Apostol: Math.'l Analysis pp. 103-107].

21. Differential 1-form as the linear approximation of a function [MTW §9.4]; Differential p-forms: antisymmetric tensor field of rank $\binom{0}{p}$; exterior product of two forms; exterior derivative of a p-form [MTW §4.1, §14.5, Ex. 14.5].

22. Parallel transport between tangent spaces [MTW Box 10.2, 10.3]; covariant differential of a vector [MTW §14.5]; covariant derivative [MTW §10.3].

23. Parallel transport in the Euclidean plane.

24. Commutator of O.N. polar basis vectors; covariant derivative of a general vector and a general covector [MTW §10.4]. Commutator vs. covariant derivative; pointwise linearity of a tensor map [MTW Box 10.3B]; parallel vector field [MTW Box 10.2], the torsion tensor [MTW §14.5 pp. 353-354]. [Note MTW always assume zero torsion.]

25. Cartan's 1st Structural equation; Stoke's Theorem: its infinitesimal version [MTW §14.5]; Riemann Curvature: Parallel transport around a closed loop [MTW §11.14]. Cartan's 2nd Structural Equation [MTW §14.5]; components of Riemann relative to a coordinate basis [MTW §11.3].

26. Jacobi's equation of geodesic deviation [MTW §11.3, Box 11.4].

27. Summary: Cartan's derivation of his two structure equations [MTW §14.5]; compatibility between mietric and parallel transport [MTW §13.3, §14.5]; Christoffel symbols; geodesic on a two sphere; coordinate components vs. orthonormal components of a vector and a covector.

28. Metric induced symmetries of the curvature tensor [MTW §11.6, 14.5].

29. Curvature from a metric: the Cartan-Misner procedure [MTW Box 14.5]. Example: the 2-sphere. Ricci tensor; Curvature Invariant.

30. Metric on a three-sphere; the metric for a Robertson-Walker universe; curvature inside spherical star in free-fall collapse; its Einstein tensor [MTW Box 14.5].

Resource Texts

1. C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation*, (Freeman, San Francisco, 1973).
2. V.I. Arnold, *Mathematical Methods of Classical Mechanics*, (Springer, 1978).
3. I.M. Singer and Thorpe, *Lecture Notes on Elementary Topology and Geometry.*
4. N.J. Hicks, *Notes on Differential Geometry*, (Van Norstround, Princeton, N.J., 1964.)
5. E.F. Taylor and J.A. Wheeler, *Spacetime Physics.*, SECOND EDITION (1st Edition)
6. T. Apostol, *Mathematical Analysis.*