The Dual of a Vector Space: From the Concrete to the Abstract to the Concrete (In Four Lectures)

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1 Motivation

Lecture 12

What facts of reality give rise to the concept "the dual of a given vector space"? Why do we need such a concept?

I. Consider the familiar circumstance, the set of fruit inventories in a supermarket. Designate a typical inventory of α^a apples, α^b bananas, α^c coconuts, $\cdots$ by

$$\vec{x} = \alpha^a \, \overrightarrow{apples} + \alpha^b \, \overrightarrow{bananas} + \alpha^c \, \overrightarrow{coconuts} + \cdots \tag{1}$$

or, more succinctly, by

$$\vec{x} = \alpha^a \vec{e}_a + \alpha^b \vec{e}_b + \alpha^c \vec{e}_c + \cdots \tag{2}$$

Fruit inventories like these form a vector space, V, closed under addition and scalar multiplication.

Nota bene: In this vector space there is no Pythagorean theorem, no distance function, and no angle between any pair of inventories. This is because in the present context apples, bananas, and coconuts are in-commensurable, i.e. "one does not mix apples and bananas".

II. Next consider a particular purchase price function, $\$_f$. Its values yield the cost of any fruit inventory $\vec{x}$:

$$\begin{aligned} \$_f(\vec{x}) &= \$_f(\alpha^a \vec{e_a} + \alpha^b \vec{e_b} + \alpha^c \vec{e_c} + \cdots) && (3) \\ &= \alpha^a \underbrace{\$_f(\vec{e_a})}_{\substack{\text{purchase} \\ \text{price/apple}}} + \alpha^b \underbrace{\$_f(\vec{e_b})}_{\substack{\text{purchase} \\ \text{price/banana}}} + \alpha^c \underbrace{\$_f(\vec{e_c})}_{\substack{\text{purchase} \\ \text{price/coconut}}} + \cdots && (4) \end{aligned}$$

Also consider the cost of fruit inventory $\vec{y}$:

$$\$_f(\vec{y}) = \beta^a \, \$_f(\vec{e_a}) + \beta^b \, \$_f(\vec{e_b}) + \beta^c \, \$_f(\vec{e_c}) + \cdots \; . \tag{5}$$

We see that

$$\$_f(\vec{x} + c\vec{y}) = \$_f(\vec{x}) + c\$_f(\vec{y}) \tag{6}$$

Thus

$$\begin{aligned} \$_f : \;\; & V \to R \\ & \vec{x} \rightsquigarrow \$_f(\vec{x}) \end{aligned}$$

is a *linear* function on the vector space V of fruit inventories.

III. The set of purchase price functions forms a vector space. Indeed, consider the purchase price functions $\$_f, \ \$_g, \ \$_h, \ \cdots$ of different fruit wholesalers, and introduce the *combined purchase price function* $\$_f + \$_g$ by the requirement that

$$(\$_f + \$_g)(\vec{x}) = \$_f(\vec{x}) + \$_g(\vec{x}) \;\; \text{for all } \vec{x} \text{ in } V \tag{7}$$

and $c\$_f$, the c-multiple of $\$_f$, by

$$(c\$_f)(\vec{x}) = c \, (\$_f(\vec{x})) \;\; .$$

We infer that the set of purchase price functions forms a vector space, V^*, the space *dual* to V.

2 Linear Functions

The space of duals is variously referred to as
the space of linear functions

the space of linear functionals
the space of covectors
the space of duals.

The concept of a dual is a new concept. It combines two concepts into one. It is a marriage between the concept of a function and the concept of a vector. Such a marriage is possible only for linear functions, a fact formalized by Theorem 1 on page 6.

In the hierarchy of concepts a dual is a derived concept, it depends on the existence and knowledge of the entities that make up a vector space. A dual conceptualizes a measurable property of these entities. For example, if one introduces a basis for the vector space, then each of the associates coordinate functions is a dual. This fact is depicted in Figure 2 on page 9. Each one is a measurable property of a vector, with the relevant basis vector serving as the relevant measurement standard. Properties such as these, and others, are mathematized by means of linear functions which are identified by the following

Definition 1 *(Linear Function)*
Let V be a vector space. Consider a scalar-valued linear function *f defined on V as follows:*

$$\begin{aligned} f: \quad & V \to R \\ & x \rightsquigarrow f(x) \end{aligned}$$

such that

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \text{ where } x, y \in V; \;\; \alpha, \beta \in \{scalars\} \; .$$

Example 1: (Linear function as a row vector)
Let $V = R^n = \{(x_1, x_2, \cdots, x_n)\}$ then

$$\begin{array}{lcccl} f: & R^n & \to & R & \\ & \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} & \rightsquigarrow & f(x_1, \cdots, x_n) & \equiv [\xi_1, \cdots, \xi_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ & & & & = \xi_1 x_1 + \cdots + \xi_n x_n \end{array}$$

is a linear function whose domain is $V = R^n$, and whose form is determined by a given row vector $[\xi_1, \cdots, \xi_n]$. See Figure 1. **Example 2:** (Weighted Sum

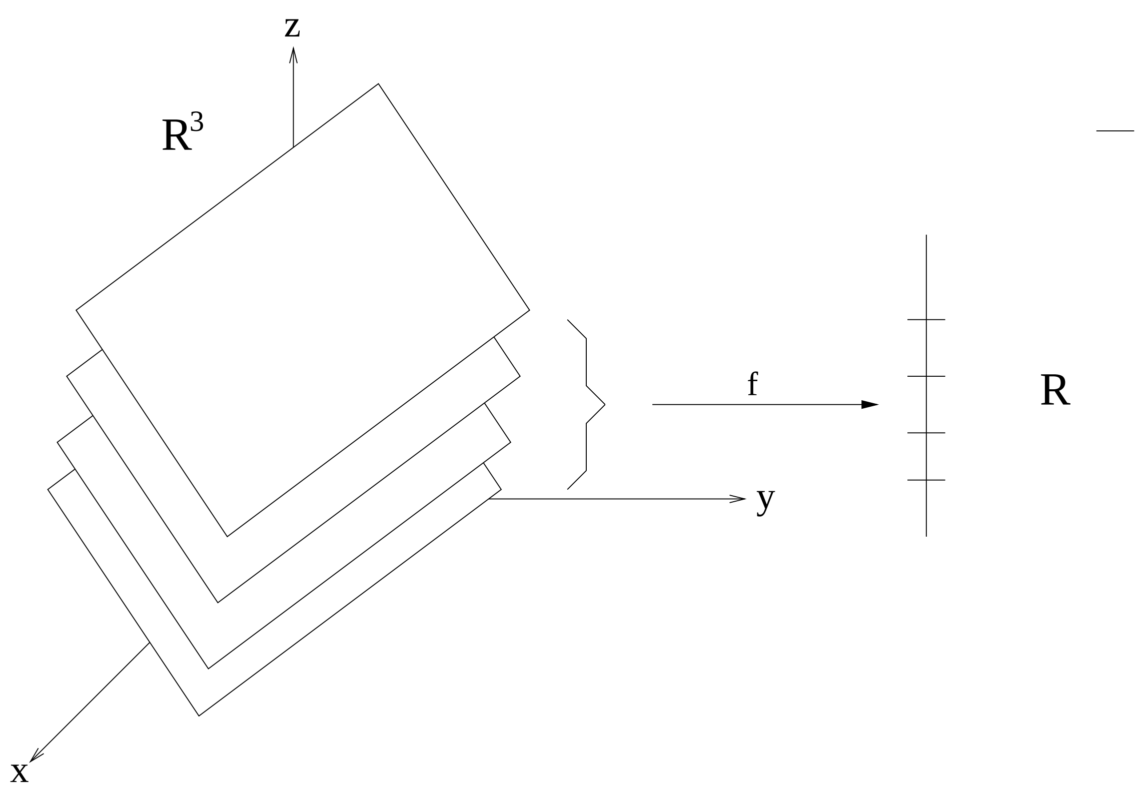

Figure 1: Linear map f. Its *isograms* (=loci of point where f has constant value) are parallel planes in its domain, R^3, with $f = 0$ passing through the origin. The image of these planes are points in R

of Samples) Consider $V = C[a,b]$, the vector space of functions continuous on the closed interval $[a,b]$:

$$V = \{\psi : \psi(s) \text{is continuous on} [a,b]\} \equiv C[a,b] \ .$$

Consider the following linear scalar-valued functions on V:

1. For any point $s_1 \in [a,b]$ the "s_1-evaluation map" (also known as the "s_1-sampling function") f

$$\begin{array}{rl} V = C[a,b] & \xrightarrow{f} R \qquad\qquad \text{(reals)} \\ \psi & \rightsquigarrow f(\psi) = \psi(s_1) \end{array}$$

   The function f is *linear* because

$$f(c_1\psi + c_2\phi) = c_1 f(\psi) + c_2 f(\phi) \tag{8}$$

2. Let $\{s_1, s_2, \cdots, s_n\} \subset [a,b]$ be a specified collection of points in $[a,b]$, and let $\{k_1, k_2, \cdots, k_n\} \equiv \{k_j\}_{j=1}^n$ be a set of scalars. Then the function g defined by

$$\begin{array}{rl} C[a,b] & \xrightarrow{g} R \\ \psi & \rightsquigarrow g(\psi) = \sum_{j=1}^n k_j \psi(s_j) \end{array}$$

   is a linear function on $V = C[a,b,]$

3. Similarly, the map h defined by

$$h(\psi) = \int_a^b \psi(s)\, ds$$

   is also a linear map on $v = C[a,b]$.

Example 3:
Consider the vector space of infinitely differentiable functions,

$$V = \{\psi : \psi \text{ is } C^\infty \text{ on } (a,b)\} \equiv C(a,b)$$

on (a,b). Furthermore, let

$$d^j\psi(s) = \left.\frac{d^j\psi}{ds^j}\right|_{x=s}$$

be the j^{th} derivative of ψ at $x = s$. Then for any *fixed* $s \in (a,b)$,the map h defined by

$$\begin{array}{rl} V = C[a,b] & \xrightarrow{h} R \\ \psi & \rightsquigarrow h(\psi) = \sum_{j=1}^n a_j d^j\psi(s) \end{array}$$

is a linear function on $V = C^\infty(a,b,)$

3 The Vector Space V^* Dual to V

Given a vector space V, the consideration of all possible linear functions defined on V gives rise to

$$V^* = \text{ set of all linear functions on V } .$$

These linear functions form a vector space in its own right, the *dual space* of V. Indeed, we have the following

Theorem 1
The set V^* is a vector space.

Comment and proof:

1. In physics the elements of the vector space V^* are called *covectors*.

2. That V^* does indeed form a vector space is verified by observing that the collection of linear functions satisfies the familiar ten properties of a vector space.

   Thus, if f, g, h are linear functions and $\alpha, \beta \in R$, then

   (a) $f + g$ is also a linear function defined by the formula

   $$(f + g)(\vec{x}) = f(\vec{x}) + g(\vec{x}) \quad \forall \vec{x} \in V \ .$$

   Consequently, the following concomitant properties are satisfied automatically:

   (b) $f + g = g + f$

   (c) $f + (g + h) = (f + g) + h$

   (d) the zero element (=“additive identity”) is the constant zero function

   (e) the additive inverse of f is $-f$.

   Furthermore,

   (i) αf is also a linear function defined by the formula

   $$(\alpha f)(\vec{x}) = \alpha f(\vec{x}) \quad \forall \vec{x} \in V \text{ and } \alpha \in R$$

   In light of this formula the following properties are also satisfied automatically:

(ii) $\alpha(\beta f) = (\alpha\beta)f$

(iii) $1f = f$

(iv) $\alpha(f + g) = \alpha f + \alpha g$

(v) $(\alpha + \beta)f = \alpha f + \beta f$

3. The two properties (a) and (b) comprise the essential, i.e. defining a.k.a. distinguishing properties of the/any vector space. They are fundamental. For conceptual economy they can be consolidated into the single statement "V^* *is closed onder linear combinations*". Stated algebraically, one has

   $$(\alpha f + \beta g)(\vec{x}) = \alpha f(\vec{x}) + \beta g(\vec{x})$$

   By contrast properties (b)-(e) and (ii)-(v) are "constitutive" properties of the vector space. It is important to remember that when considering any concept, that concept includes *all* of its properties, not some to the exclusion of others.

4 Dirac's Bracket Notation

To emphasize the duality between the two vector spaces, one takes advantage of Dirac's *bra-ket notation*, which he originally introduced into quantum mechanics.

If f is a linear function on V and $f(x)$ is its value at $x \in V$, then one also writes

$$f(x) \equiv \langle f|x\rangle \equiv \langle \underline{f}|\vec{x}\rangle \tag{9}$$

Thus the underscore under f is a reminder that $\underline{f} \in V^*$, while x, or better $\vec{x}$ is an element of V. We say that f *operates* on the vector x and produces

$$\langle f|x\rangle$$

To emphasize that f is a linear "machine", we write

$$f = \langle \underline{f}| \quad (\in V^*) \tag{10}$$

for the covector (which Dirac called a bra) and

$$x = |\vec{x}\rangle \quad (\in V) \tag{11}$$

for the vector (which Dirac called a ket). They combine to form

$$\langle \underline{f} | \vec{x} \rangle \quad (\in R)$$

5 The Duality Principle

Lecture 13

As we shall see, mathematically it is the existence and uniqueness of a vector's scalar coefficients relative to a chosen/given basis that makes the concept of duality so important. Indeed, such a basis makes the introduction of V^* inevitable. This is because a basis determines unique scalar values for each vector, which is to say that it determines scalar functions on V.

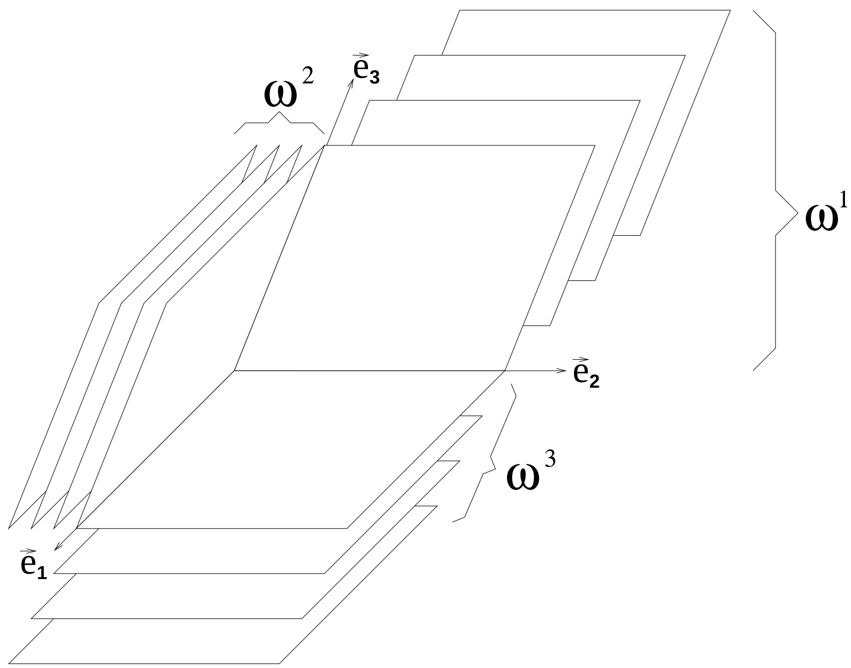

Figure 2: Level surfaces (="isograms") of the coordinate functions, which (i) are linear on V and (ii) are determined by the basis $\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n$. The tip of a vector (not shown here), say, $x = \alpha_1\vec{e}_1 + \alpha_2\vec{e}_2 + \alpha_3\vec{e}_3$, is located at the intersection of three isograms. This is the point where $\omega^1(x) = x^1$, $\omega^2(x) = x^2$, and $\omega^3(x) = x^3$.

The problem, therefore, is: What are these scalar functions? Are they elements of V^*? If so, do they form a linearly independent set? Do they span V^*?

The answer to these questions gives rise to

The Duality Principle:
For each ordered basis

$$\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$$

of a finite dimensional vector space V, there exists a corresponding basis

$$\{\underline{\omega}^1, \underline{\omega}^2, \cdots, \underline{\omega}^n, \}$$

for V^*, and vice versa, such that

$$\langle \underline{\omega}^i | \vec{e}_j \rangle = \delta^i_j \ . \tag{12}$$

Warning. The evaluation, Eq.(12), is not to be confused with an inner product. The existence of the duality between V and V^* by itself does not at all imply the existence of an inner product. We shall see that the existence of an inner product on a vector space establishes a unique basis-independent (= "natural") isomorphic correspondence between V and V^*. In the absence of an inner product such a correspondence does not exist.

The validation of the duality principle consists of the actual three-step construction of the basis dual to the given basis, which we denote by

$$B = \{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\} \subset V \quad \text{(basis for } V.)$$

Step I.
For all vectors x and y one has the following unique expansions:

$$\begin{aligned} x &= \alpha^1 \vec{e}_1 + \cdots + \alpha^n \vec{e}_n \\ y &= \beta^1 \vec{e}_1 + \cdots + \beta^n \vec{e}_n \\ x + y &= (\alpha^1 + \beta^1)\vec{e}_1 + \cdots + (\alpha^n + \beta^n)\vec{e}_n \\ cx &= c\alpha^1 \vec{e}_1 + \cdots + c\alpha^n \vec{e}_n \qquad (c \text{ is a scalar}) \end{aligned} \tag{13}$$

Note that

$$\begin{array}{rccccc} \alpha^1 & \text{is} & \text{uniquely} & \text{determined} & \text{by} & x \\ \beta^1 & '' & '' & '' & '' & y \\ \alpha^1 + \beta^1 & '' & '' & '' & '' & x + y \\ c\alpha^1 & '' & '' & '' & '' & cx \end{array}$$

Step II.
These four relations determine a <u>linear function</u>, call it ω^1. Its defining prop-

erties are

$$\begin{aligned}
\omega^1(x) &= \alpha^1 \\
\omega^1(y) &= \beta^1 \\
\omega^1(x+y) &= \alpha^1 + \beta^1 \\
\omega^1(cx) &= c\alpha^1
\end{aligned}$$

which imply

$$\begin{aligned}
\omega^1(x+y) &= \omega^1(x) + \omega^1(y) \\
\omega^1(cx) &= c\omega^1(x)
\end{aligned}$$

In particular, using Eq.(13) on page 10, one has

$$\begin{aligned}
\omega^1(\vec{e}_1) &= 1 \\
\omega^1(\vec{e}_2) &= 0 \\
&\vdots \\
\omega^1(\vec{e}_n) &= 0.
\end{aligned}$$

We conclude that ω^1 is a linear function, indeed. The function ω^1 is called the first coordinate function.

Step III.

Similarly the j^{th} coordinate function, is defined by

$$\omega^j = \alpha^j \quad \text{for } j = 2, 3, \cdots, n \ .$$

By applying ω^j to the ith basis vector $\vec{e}_i$, and using Eq.(13) on page 10 one obtains

$$\omega^j(\vec{e}_i) \equiv \langle \omega^j | \vec{e}_i \rangle = \left\{ \begin{array}{ll} 1 & j = i \\ 0 & j \neq i \end{array} \right.$$

or in terms of the Kronecker delta,

$$\boxed{\langle \omega^j | \vec{e}_i \rangle = \delta^j{}_i} \ .$$

This is called a duality relation or duality principle. The choice of a different vector basis would have resulted in a correspondingly different set of coordinate functions, but would have again resulted in a duality relation.

Being elements in V^*, do these coordinate functions form a basis for V^*? The answer to this important question is answered in the affirmative by the following

Theorem 2 *(Dual Basis)*

Given: *A basis* $B = \{\vec{e}_1, \cdots, \vec{e}_n\}$ *for* V*.*

Conclusion: *The set of linear functions* $B^* = \{\omega^j\}_{j=1}^n$ *which satisfies the duality relation*

$$\langle \omega^j | \vec{e}_i \rangle = \delta^j{}_i \tag{14}$$

is a <u>*basis*</u> *for* V^**.*

The proof of the spanning property of B^* hinges on the spanning property of B as follows: Let $f \in V^*$ be some linear function on V. Evaluate $f(x)$ and use $x = \sum_i \alpha^i \vec{e}_i$. Thus

$$\begin{aligned} f(x) &= f\left(\sum_i \alpha^i \vec{e}_i\right) \\ &= \sum_i f(\vec{e}_i)\, \alpha^i \qquad \alpha^i \text{ is the i}^{th} \text{ coord. of } x, \text{ i.e. } \alpha^i = \omega^i(x); \\ &= \sum_i f(\vec{e}_i)\, \omega^i(x) \qquad \forall x \in V \end{aligned}$$

This holds for all $x \in V$. Consequently,

$$f = \sum_i f(\vec{e}_i)\, \omega^i, \tag{15}$$

which is an expansion of f in terms of the elements of B^*, which means that B^* is a spanning set for V^* indeed.
To show that B^* has the linear independence property, we consider the equation

$$c_1\omega_1 + c_2\omega_2 + \cdots + c_n\omega_n = \underline{0}$$

where $\underline{0}$ is the function with constant value zero on V. By evaluating both sides on the ith basis vector $\vec{e}_i$ and using Eq.(14) one obtains

$$c_i = 0 \qquad \text{for } i = 1, 2, \cdots, n$$

Consequently, B^* does have the linear independence property. Together with its spanning property, this validates the claim made in the Theorem that B^* is a basis for V^*.
Example 1 (Column space*=Row space)
GIVEN:
Let

$$B = \left\{ \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \vec{e}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \vec{e}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

be a basis for the column space $V = R^3$.
a) IDENTIFY V^*, the space dual to V.
b) FIND the basis $B^* = \{\omega_1; \omega_2; \omega_3\}$ dual to B, i.e. exhibit elements ω^j which satisfy Eq.(14).
Solution
a) The space dual to V consists of the *row space*

$$V^* = \{\sigma = [a\, b\, c] : \ a, b, c \in R\}.$$

Indeed, for any $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in R^3$

$$\begin{aligned} \langle \sigma | x \rangle &\equiv \sigma \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) \\ &= [a\ b\ c] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax + by + cz \end{aligned}$$

Question:
What line of reasoning led to the fact that the answer to a) is the space of *row* vectors?
Answer:
A prerequisite for the course “Linear Mathematics in Finite Dimensions” is a knowledge of matrix theory (as, for example, in chapter 1 of Johnson, Riess, and Arnold). One of the concepts in this chapter is that of a *row vector*. One of the constitutive properties of a row vector is that it can be multiplied by a column vector and thereby produces a scalar number. This property was

stored in our subconscious (the "hard disk" of our consciousness) where it has been ever since.
Now part a) of the above problem, asks for linear functions on the space of column vectors. That is the standing order, to search our subconscious for a concept with this requisite property. The success of this search was not immediate. In fact, we had to "sleep" on it. However, the already-known concept "row vector" is the *green light* to inferring generalizations from particular instances. In particular, a row vector, when applied to column vectors, produces scalar numbers. The generalization is "row vectors give rise to (i.e. imply) linear functions". This inference is mandatory for two reasons: (i) a function is precisely the process of assigning scalar values to elements, here column vectors in the function's domain and (ii) the process is a linear one.

This generalization is a causal relation between row vectors and linear functions. Like all generalizations it is new knowledge. We arrived at it not deductively ("All men are mortal; Socrates is a man; hence Socrates is mortal") , but by the process of induction. This process is much more difficult and requires much more effort because it involved *all* our relevant knowledge, namely matrix theory.

To state it negatively and more generally: generalizations are not obtained by "intuition", "inspiration", "revelation", or by some other kind of pseudo explanation. Instead the road to success is paved by hard work together with by not letting ones subconscious "goof off", but giving it a standing order(s) consisting of valid concepts.
b) Each ω^j is a row vector. They must satisfy

$$\left.\begin{aligned}
\langle\omega^1|\vec{e}_1\rangle &= [a\;b\;c]\begin{bmatrix}1\\0\\0\end{bmatrix} = 1\\
\langle\omega^1|\vec{e}_2\rangle &= [a\;b\;c]\begin{bmatrix}1\\1\\0\end{bmatrix} = 0\\
\langle\omega^1|\vec{e}_3\rangle &= [a\;b\;c]\begin{bmatrix}1\\1\\1\end{bmatrix} = 0
\end{aligned}\right\} \Rightarrow \omega^1 = [1\;{-1}\;0] \tag{16}$$

$$\left.\begin{aligned}
\langle\omega^2|\vec{e}_1\rangle &= [d\,e\,f]\begin{bmatrix}1\\0\\0\end{bmatrix} = 0\\
\langle\omega^2|\vec{e}_2\rangle &= [d\,e\,f]\begin{bmatrix}1\\1\\0\end{bmatrix} = 1\\
\langle\omega^2|\vec{e}_3\rangle &= [d\,e\,f]\begin{bmatrix}1\\1\\1\end{bmatrix} = 0
\end{aligned}\right\} \Rightarrow \omega^2 = [0\;1\;{-1}] \tag{17}$$

$$\left.\begin{aligned}
\langle\omega^3|\vec{e}_1\rangle &= [u\,v\,w]\begin{bmatrix}1\\0\\0\end{bmatrix} = 0\\
\langle\omega^3|\vec{e}_2\rangle &= [u\,v\,w]\begin{bmatrix}1\\1\\0\end{bmatrix} = 0\\
\langle\omega^3|\vec{e}_3\rangle &= [u\,v\,w]\begin{bmatrix}1\\1\\1\end{bmatrix} = 1
\end{aligned}\right\} \Rightarrow \omega^3 = [0\;0\;1] \tag{18}$$

Thus the basis of duals for V^*, the space dual to $V = R^3$ is

$$B^* = \{\omega^j\}_{j=1}^3 = \{\ [1\;{-1}\;0],\ [0\;1\;{-1}],\ [0\;0\;1]\ \}$$

Example 2
Same as Example 1 on page 13, except that

$$B = \left\{\vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix};\vec{e}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix};\vec{e}_3 = \begin{bmatrix}1\\1\\2\end{bmatrix}\right\}$$

Answer:

$$B^* = \{\omega^j\}_{j=1}^3 = \{\ [1\;{-1}\;\frac{1}{2}],\ [0\;1\;{-1}],\ [0\;0\;\frac{1}{2}]\ \}$$

Comment.
Note that changing only one element of B, say, $\vec{e}_j \to \vec{e}\,'_j$ changes several elements of B^*, as in Figure 3 on page 16. This implies that there is as-yet no basis independent correspondence between V^* and V. A basis independent correspondence would have required that changing only one of the basis

vectors in V would have produced a corresponding change in only one basis vector in V^*.
Summary.
There does exist a unique correspondence between ordered basis sets in V and V^*

$$\{\vec{e}_i\}_{i=1}^n \leftrightarrow \{\omega^j\}_{j=1}^n,$$

but not between individual vectors in V and V^*:

$$\{\text{coordinate vectors}\} \not\leftrightarrow \{\text{coordinate surfaces}\}$$

More succinctly, one says that there exists no natural (i.e. basis independent isomorphism between V and V^*.

However, we shall see in the next lecture that if V is endowed with an *inner product*, then *there is* a natural isomorphism between V and V^*.

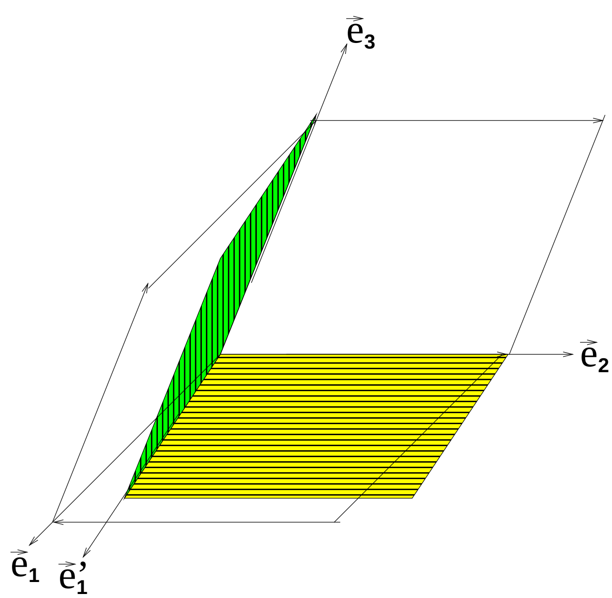

Figure 3: Changing only one vector of a basis, for example $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \longrightarrow \{\vec{e}_1', \vec{e}_2, \vec{e}_3\}$, changes more than one dual element as evidenced by the tilting of their coordinate surfaces in the vector space.

6 Geometry of Duals in an Oblique Cordinate System

An element of V^*, a covector, is a linear function on V. Its basis expansion, say,

$$f = \xi_1\omega^1 + \xi_2\omega^2 + \cdots + \xi_n\omega^1,$$

mathematized the nature of its level surfaces. With the given/chosen basis

$$B = \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$$

for V, these levels surfaces have a *measurable density of isograms.* Thinking in terms of f as the linear phase function of a plane wave, one refers to these isograms as phase fronts which are parallel. The basis vectors $\mathbf{e}_1, \cdots, \mathbf{e}_n$ serve as the standard of measurent. The physical and geometrical meaning of the expansion coefficients $\xi_1, \cdots, \xi_n$ is summarized by the following
Definition

$$\begin{aligned} \xi_1 = f(\mathbf{e}_1 &= \textit{density of isograms } (\text{``}\textit{phasefronts}'')\text{into the } \mathbf{e}_1 - \text{direction} \\ &\vdots \\ \xi_n = f(\mathbf{e}_n &= \textit{density of isograms } (\text{``}\textit{phasefronts}'')\text{into the } \mathbf{e}_n - \text{direction} \end{aligned}$$

Thus the density of the isograms of $f \in V^*$ consists of its n components $\{\xi_i : i = 1, \cdots, n\}$, the density components measured relative to the basis B, which is the chosen standard of measurements.

Figure 4 illustrates this definition for the two-dimensional covector

$$f = 2\omega^1 + \omega^2.$$

Its densities into the direction of the basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are

$$\begin{aligned} f(\mathbf{e}_1) = \langle f|\mathbf{e}_1\rangle = 2 \\ f(\mathbf{e}_2) = \langle f|\mathbf{e}_2\rangle = 1 \end{aligned} \tag{19}$$

Once one has agreed on a basis as the chosen standard of measurement, these densities determine and are determined by that cofactor/linear function.
Q: Where in V are the (integral-valued) isograms of f?
A: From Eq.(19) and the linearity of f one obtains

$$\begin{aligned} \langle f|\tfrac{1}{2}\mathbf{e}_1\rangle = 1 \\ \langle f|\mathbf{e}_2\rangle = 1 \end{aligned} \tag{20}$$

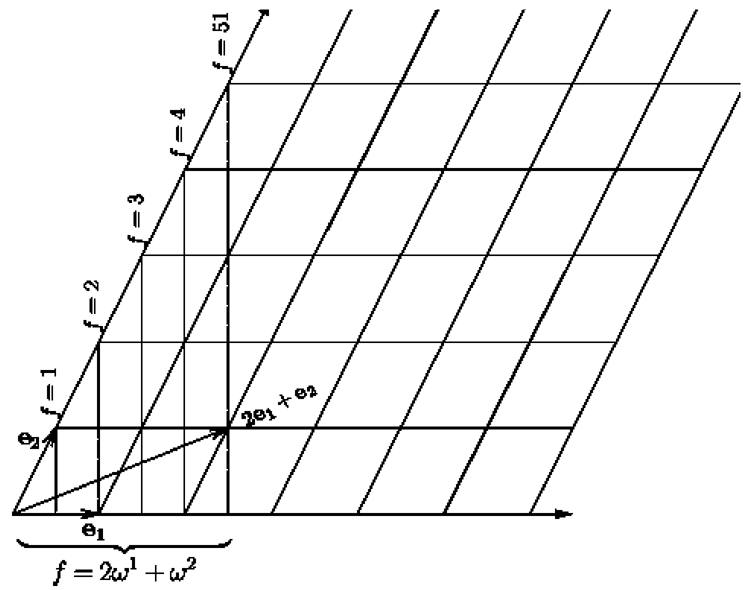

Figure 4: Vector $2\mathbf{e}_1 + \mathbf{e}_2$ piercing five integral isograms of $2\omega^1 + \omega^2$.

Thus the $f = 1$ isogram passes through the two vectors $\frac{1}{2}\mathbf{e}_1$ and $\mathbf{e}_2$. All the other isograms – including the $f = 0$ isogram (not shown in the figure), which passes through the origin – are parallel to that isogram. Figure 4 depicts the physical relationship between f with its isograms on one hand, and some vector on which f is evaluated on the other:

$$\langle f|2\mathbf{e}_1 + \mathbf{e}_2\rangle = \langle 2\omega^1 + \omega^2|2\mathbf{e}_1 + \mathbf{e}_2\rangle = 4 + 0 + 0 + 1 = 5$$

From this figure one sees that the vector $2\mathbf{e}_1 + \mathbf{e}_2$ pierces 5 integral-valued isograms of $2\omega^1 + \omega^2$.

7 Metric as a Bilinear Function on a Vector Space

Lecture 14

The vector space arenas developed so far are in skeleton form but fundamental to all of mathematics. In physics and engineering terminology their linearity is captured by means of the *superposition principle*. In mathematics, by means of *closure under linear combination*.

The bare bones attributes introduced so far are the *linear (in)dependence* and the *spanning property* of a set of vectors. These properties are sufficient for characterizing a vector space in terms of coordinate systems introduced via any chosen (or given) basis. As a result every vector space V accommodates its dual space V^*, the space of linear functions on V. This space is a vector space in its own right, and any basis for V determines a unique corresponding basis for V^*. Indeed, the dimensions of V and V^* are the same, a fact which is a consequence of the duality principle

$$\langle \underline{\omega}^j | \vec{e}_j \rangle = \delta^j{}_i$$

In spite of this duality, there is no natural (i.e. basis-independent) correspondence between V and its dual space of covectors, V^*.

This deficiency, as we shall see, disappears once one has identified an inner product on the given vector space arena.

7.1 Bilinear Functional; the Metric

There is no natural isomorphism between V and V^*. However, if the vector space has an inner product defined on it, then such an isomorphism is determined.

Definition 2 *(Bilinear Form)*
Given: a vector space U and a vector space V.
A bilinear functional *(or* "form"*) on $U \times V$ (= pairs of elements, one from U and one from V) is a function w,*

$$\begin{array}{rl} w: & U \times V \stackrel{w}{\to} \textit{scalars} \\ & (x, y) \rightsquigarrow w(x, y) \end{array}$$

with the properties

$$w(\alpha^1 x_1 + \alpha^2 x_2, y) = \alpha^1 w(x_1, y) + \alpha^2 w(x_2, y)$$
$$w(x, \beta^1 y_1 + \beta^2 y_2) = \beta^1 w(x, y_1) + \beta^2 w(x, y_2);$$

in other words, w is linear in each argument.

In this definition U and V can be vector spaces of different dimensions. The concept of a metric arises if the two vector spaces are one and the same and is given by the following definition:

Definition 3 *(Metric) A* metric *(or* inner product*) is a bilinear functional g on $V \times V$ (pairs of elements in V)*

$$g: \ (x, y) \ \rightsquigarrow \ \ g(x, y)$$

with the property

$$g(x, y) = g(y, x) \ .$$

In other words, a real-valued metric is symmetric.
Comment.
If the metric were complex-valued, then the symmetry condition get replaced by

$$g(x, y) = \overline{g(y, x)}$$

The metric $g(\ , \ )$ is said to be an *inner product* whenever g is positive definite, i.e. $g(x, x) > 0 \ \forall x \neq 0$.
Example (Basis Expansion of the Metric) Let

$$x = x^1 \vec{e}_1 + x^2 \vec{e}_2 + \cdots + x^n \vec{e}_n$$

be a representation of a vector x relative to a basis $\{\vec{e}_1, \cdots, \vec{e}_n\}$ for V. Then

$$\begin{aligned}
g(x, y) &= g(x^1 \vec{e}_1 + x^2 \vec{e}_2 + \cdots + x^n \vec{e}_n, y^1 \vec{e}_1 + y^2 \vec{e}_2 + \cdots + y^n \vec{e}_n) \\
&= x^1 y^1 g(\vec{e}_1, \vec{e}_1) + (x^1 y^2 + x^2 y^1 g(\vec{e}_1, \vec{e}_2) + x^2 y^2 g(\vec{e}_2, \vec{e}_2) + \cdots \\
&= x^1 y^1 \vec{e}_1 \cdot \vec{e}_1 + (x^1 y^2 + x^2 y^1) \vec{e}_1 \cdot \vec{e}_2 + x^2 y^2 \vec{e}_2 \cdot \vec{e}_2 + \cdots \\
&= x^1 y^1 g_{11} + (x^1 y^2 + x^2 y^1) g_{12} + x^2 y^2 g_{22} + \cdots \\
&= x^i y^j g_{ij} \quad \text{(Einstein summation convention for pairs of repeated indeces)}
\end{aligned}$$

The coefficients $g_{ij} \equiv \vec{e}_i \cdot \vec{e}_j \equiv g(\vec{e}_i, \vec{e}_j)$ are the *components* of the metric g *relative* to the given basis. They are the innerproducts of all pairs of basis vectors.

7.2 Metric as an Isomorphism between Vector Space and its Space of Duals

The scalar product

$$\begin{aligned} g: \quad V \times V &\rightarrow R \\ (x, y) &\rightsquigarrow g(x, y) \end{aligned} \tag{21}$$

is a bilinear function. Consequently, it can be evaluated on only one of its arguments, $g(x, \ )$.The result is a linear function. More precisely, a metric establishes a natural (i.e. basis independent) isomorphism between vector space V and its space of duals, V^*. In order to conserve notation we shall use the same symbol g to designate this correspondence. Its defining property is

$$\begin{aligned} g: \quad V &\rightarrow V^* \\ x &\rightsquigarrow g(x, \ ) \equiv \underline{x} \ (= x\cdot) \end{aligned} \tag{22}$$

Here $\underline{x}$ is that linear function which, when evaluated on $y \in V$, yields $g(x, y)$:

$$\begin{aligned} \underline{x} = x\cdot : \quad V &\overset{x}{\rightarrow} V^* \\ y &\rightsquigarrow \langle \underline{x} | y \rangle = x \cdot y \equiv g(x, y) \end{aligned}$$

If g maps $\vec{x}$ to its image $\underline{x}$, what is the image of the set of components of $\vec{x}$? The answer is given by the following proposition,

Proposition

> Given the vector $\vec{x} = x^k \vec{e}_k$, the numerical coefficients x_j of the corresponding covector $\underline{x} = x_j \underline{\omega}^j$ are given explicitly by the following computation:
>
> $$\begin{aligned} \underline{x} &= g(\vec{x}, \ \ ) \\ &= x^k g(\vec{e}_k, \ ) \end{aligned}$$
>
> Taking advantage of the spanning property of $\{\omega^j\}$, Eq.(15) on page 12), we find that the to-be-determined x_j satisfy
>
> $$x^k g(\vec{e}_k, \ ) = x_j \omega^j$$

Evaluating both side on each of the basis vectors e_i, and using the duality relation Eq.(14), we obtain

$$\begin{aligned} x_i &= x^k g(\vec{e}_k, \vec{e}_i) \\ &= x^k g_{ki} \\ &\equiv \vec{x} \cdot \vec{e}_i, \end{aligned} \tag{23}$$

the components of $x_j \omega^j$, which is the image of $\vec{x} = x^k \vec{e}_k$ under g.

Equations (21) and (22) two seemingly disparate[1] aspects of the metric. However, the fact of the matter is they are merely different manifestation of the metric concept. This becomes blindingly obvious when one introduces any basis, say, $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ for V and its dual basis $\omega^1, \cdots, \omega^n\}$ coordinate functions. Relative to these bases the bilinearity of g implies that

$$g(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y} \tag{24}$$

$$= \vec{e}_i \cdot \vec{e}_j \, x^i y^j \tag{25}$$

$$= g_{ij} \, \omega^i(\vec{x}) \, \omega^j(\vec{y}) \tag{26}$$

$$\equiv g_{ij} \, \omega^i \otimes \omega^j(\vec{x}, \vec{y}) \qquad \forall \, \vec{x}, \vec{y} \in V. \tag{27}$$

Consequently,

$$\boxed{g = g_{ij} \, \omega^i \otimes \omega^j} \; .$$

Here the product symbol $\otimes$ establishes an ordered juxtaposition of two *linear* function(al)s, thereby yielding a *bi*linear function(al).
On the other hand, the linearity of $g = g_{ij} \, \omega^i \otimes \omega^j$ in each argument implies that for a given $\vec{x}$ one has

$$\begin{aligned} g_{ij} \, \omega^i \otimes \omega^j(\vec{x}, \vec{y}) &= g_{ij} \omega^i(\vec{x}) \, \omega^j(\vec{y}) \\ &= g_{ij} \, x^i \omega^j(\vec{y}) \\ &= x_j \omega^j(\vec{y}) \qquad \text{(as defined by Eq.(23) on page 22)} \\ &\equiv \underline{x} \, (\vec{y}) \qquad \forall y \in V \end{aligned}$$

Consequently,

$$\begin{aligned} g_{ij} \, \omega^i \otimes \omega^j(\vec{x}, \;\;) &\equiv g_{ij} \omega^i(\vec{x}) \, \omega^j \\ &= \underline{x} \end{aligned}$$

[1] The dictionary definition of this word is “essentially different in kind; not allowing comparison”

which is to say that $g = g_{ij}\,\omega^i \otimes \omega^j$ is a *linear* one-to-one mapping $V \to V^*$ which assigns $\vec{x} \in V$ to $\underline{x} \in V^*$.

Implicit in the metric-induced isomorphism

$$\begin{aligned} V &\xrightarrow{g} V^* \\ \vec{x} &\leadsto \underline{x} \end{aligned}$$

is the existence of a vector normal to $\underline{x}$'s isograms[2] in V. In fact, this normal is precisely the preimage of $\underline{x}$, namely $\vec{x}$ itself. Moreover, each set of isograms of each coordinate function ω^i has such a normal. This circumstance gives rise to the basis *reciprocal* to the originally given basis. This reciprocal basis mathematizes the obliqueness of the given coordinate surfaces. It is developed in the next subsection.

[2]An *isogram* (a.k.a. a level surface) of a function is the locus of points where the function has the same constant value

8 Mathematizing an Oblique Coordinate System

Lecture 15

Historically the "contravariant" components of a vector are its components relative to a chosen/given basis. Indeed, they are "the" components relative to this basis, regardless of what metric the vector space V is endowed with.

If V is endowed with a specific metric then there exists a second well-defined basis, "reciprocal" to the first one. The reciprocal basis consists of

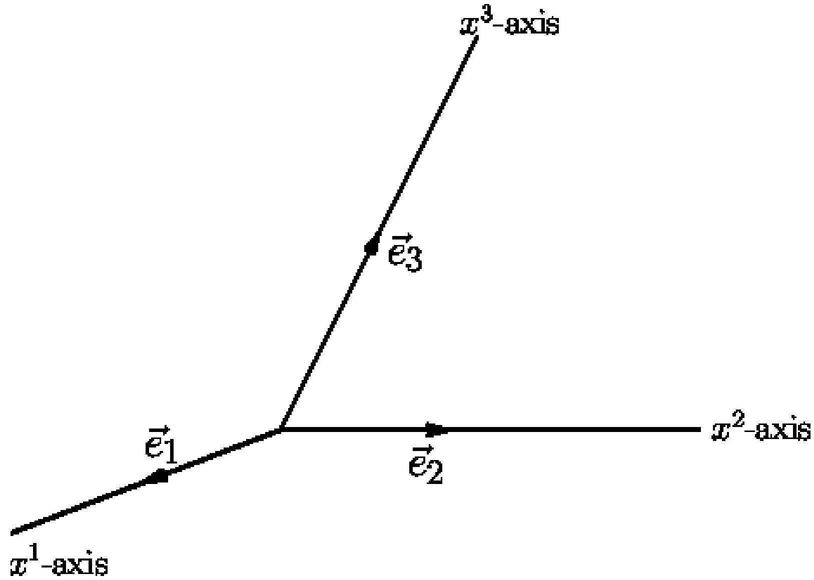

Figure 5: Oblique basis in R^3

the vectors which are perpendicular the coordinate planes spaned by the given basis vectors. As depicted in Figure 5

$\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2$ span the 1-2 plane,

$\tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$ span the 2-3 plane,

$\tilde{\mathbf{e}}_3, \tilde{\mathbf{e}}_1$ span the 1-2 plane.

The scaled vectors *perpendicular* to these planes form the *reciprocal basis*

$$\{\tilde{\mathbf{e}}_1^*, \tilde{\mathbf{e}}_2^*, \tilde{\mathbf{e}}_3^*\}.$$

They are determined by the inner product condition

$$\tilde{\mathbf{e}}_k^* \cdot \tilde{\mathbf{e}}_i = \delta_{ki} = \begin{cases} 0 & k \neq i \\ 1 & k = i \end{cases} \tag{28}$$

Warning:
The appellations "contravariant vector" and "covariant vector" are invalid concepts. They are oxymorons, examples of mixing incommensurable categories, an attempt to blend mutually exclusive ideas into a single unit. Indeed, a vector is a *basis independent* concept, while "contravariant" or "covariant" are attributes of the components of a vector and thus are *relative to some basis*.

In two dimensions the reciprocal basis and its properties arise as follows:

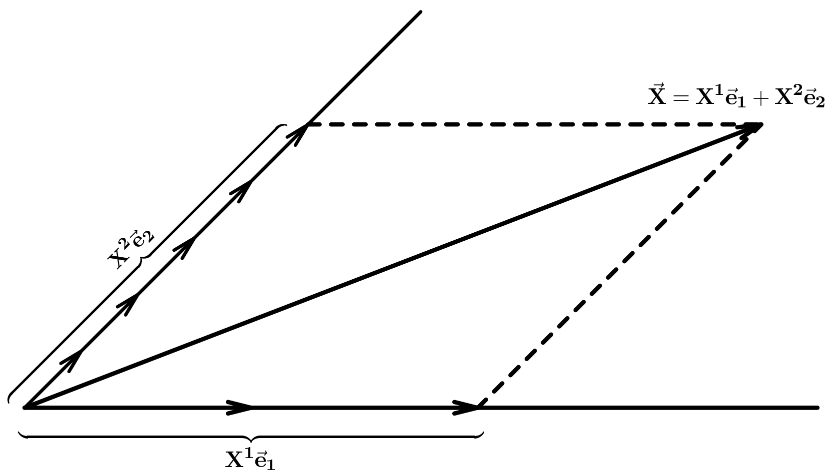

Figure 6: Oblique basis and its oblique coordinate system.

Start with an oblique basis $\{\vec{e}_1, \vec{e}_2\}$ and its coordinate system as in Fig. 6. Next introduce a vector, $\vec{e}_1^{\,*}$, which is perpendicular to $\vec{e}_2$ and is "normalized" by being reciprocal to $\vec{e}_1$ as in Figure 7 on page 27

$$\vec{e}_1^{\,*} \cdot \vec{e}_2 = 0$$
$$\vec{e}_1^{\,*} \cdot \vec{e}_1 = 1$$

In a similar way introduce $\vec{e}_2^*$ which is perpendicular to $\vec{e}_1$ and reciprocal to $\vec{e}_2$:

$$\begin{aligned} \vec{e}_2^* \cdot \vec{e}_1 &= 0 \\ \vec{e}_2^* \cdot \vec{e}_2 &= 1 \end{aligned}$$

The basis $\{\vec{e}_1^*, \vec{e}_2^*\}$ constructed in this manner is *reciprocal* to the given basis $B = \{\vec{e}_1, \vec{e}_2\}$ because it satisfies

$$\vec{e}_i^* \cdot \vec{e}_j = \delta_{ij}.$$

More generally, we have the following

Definition 4 *(Reciprocal Basis)*

Given (i) the metric $g ='' \cdot ''$, a metric on v

(ii) a basis $\{\vec{e}_1, \cdots, \vec{e}_n\}$ for V,

then the set of vectors

$$\{\vec{e}_1^*, \cdots, \vec{e}_n^*\},$$

where

$$\vec{e}_k^* \cdot \vec{e}_j = \delta_{kj}, \tag{29}$$

is the basis reciprocal *to* $\{\vec{e}_j\}_{j=1}^n$.

This definition says that $\{\vec{e}_k\}$ is a vector perpendicular to the plane containing the vectors $\{\vec{e}_1, \cdots, \vec{e}_{k-1}, \vec{e}_{k+1}, \cdots, \vec{e}_n\}$, i.e.

$$\boxed{\vec{e}_k^* \cdot \vec{e}_j = 0 \quad j \neq k.}$$

Furthermore, $\vec{e}_k^*$ is scaled such that

$$\boxed{\vec{e}_k^* \cdot \vec{e}_k = 1 \quad \text{(No sum over } k)}$$

It is also clear that if the basis $\{\vec{e}_k^*\}_{k=1}^n$ is reciprocal to $\{\vec{e}_j\}_{j=1}^n$, then $\{\vec{e}_j\}_{j=1}^n$ is reciprocal to $\{\vec{e}_k^*\}_{k=1}^n$.

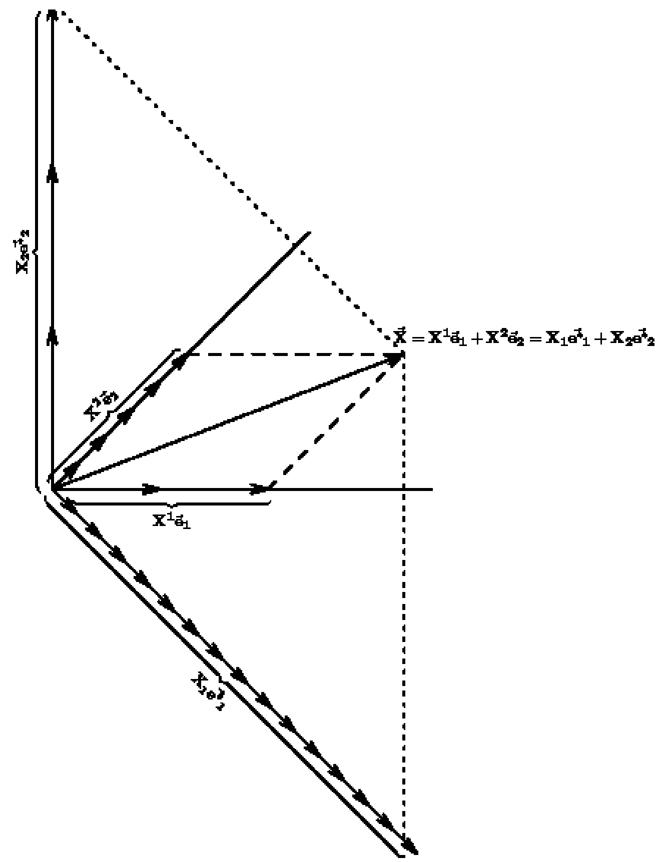

Figure 7: Oblique basis and its reciprocal basis

8.1 The Geometric Relation Between a Vector Space and its Dual

There is a fundamental relation between

(i) the basis $\{\vec{e}^{\,*}_j\}_{j=1}^n \equiv R^*$ reciprocal to the given basis $B = \{\vec{e}_i\}_{i=1}^n$, with $\vec{e}^{\,*}_j \cdot \vec{e}_i = \delta_{ij}$,
and

(ii) the basis $\{\omega^j\}_{j=1}^n = B^*$ dual to B:

The respective elements of either basis are projection operators. Both yield the coordinate components of any vector. This claim is mathematized by means of the following proposition, which, once validated, establishes a *natural* isomorphism between V and V^*.
Proposition (Projection Operators)

> The *dual basis* as well as the *reciprocal basis* serve as projection operators in that they yield the coordinates of a vector:
>
> $$\langle \omega^k | \vec{x} \rangle = \langle \omega^k | x^i \vec{e}_i \rangle = x^i \delta^k_i = x^k$$
> $$\vec{e}^{\,*}_k \cdot \vec{x} = \ x^i \vec{e}^{\,*}_k \cdot \vec{e}_i = x^i \delta_{ki} = x^k$$

This proposition is the first step in mathematizing the geometrical relation between
(1) a vector space with an (in general) oblique coordinate system and
(2) its dual space.

The second step consists of taking advantage of the fact that the metric $g(\ ,\ ) =$ "$\cdot$" is linear in its second argument. Indeed, for each of the elements $\vec{e}^{\,*}_j$ of the reciprocal basis R^* the metric implies the $\vec{e}^{\,*}_j$-parametrized linear function

$$g(\vec{e}^{\,*}_1,\ ) = \omega^1(\ \ ) \tag{30}$$
$$\vdots$$
$$g(\vec{e}^{\,*}_n,\ ) = \omega^n(\ \ ), \tag{31}$$

which, of course, are precisely the dual basis elements.

The third and final step consists of taking advantage of g's linearity in its first argument. Taking linear combination, one has

$$g\left(\sum_{k=1}^{n} \alpha_k \vec{e}^{\,*}_k,\ \right) = \alpha_k \omega^k(\ \ ). \tag{32}$$

The significance of the cluster of equalities (30)-(32) is that they allow being consolidated into the single function

$$g:\ V \longrightarrow V^*$$

whose concrete values are given by

$$
\begin{aligned}
\vec{e}_1^{\,*} &\overset{g}{\rightsquigarrow} \omega^1 \\
&\vdots \\
\vec{e}_n^{\,*} &\overset{g}{\rightsquigarrow} \omega^n \\
\sum_{k=1}^{n} \alpha_k \vec{e}_k^{\,*} &\overset{g}{\rightsquigarrow} \alpha_k \omega^k
\end{aligned}
\tag{33}
$$

$$
\text{i.e.} \qquad \underbrace{\qquad\quad}_{\vec{a}} \;\overset{g}{\rightsquigarrow}\; \underbrace{\qquad}_{\underline{a}}
\tag{34}
$$

Q: What is the geometrical meaning of Eq.(34)?
The answer is given by evaluating the covector $\underline{a}$ on a generic vector $\vec{x}$:

$$
\begin{aligned}
\underline{a}(\vec{x}) &= \langle \alpha_k \omega^k | \vec{x} \rangle \\
&= \alpha_k \omega^k (\vec{x}) \\
&= \alpha_k x^k \\
&= \sum_{k=1}^{n} \alpha_k \vec{e}_k^{\,*} \cdot \vec{x} \\
&= \vec{a} \cdot \vec{x}
\end{aligned}
\tag{35}
$$

Consider the $\underline{a}=0$ isogram of the linear function $\underline{a}$. It consists of all vectors $\vec{x}$ which satisfy

$$\underline{a}(\vec{x}) = 0.$$

In light of Eq.(35) this means that

$$\vec{a} \cdot \vec{x} = 0.$$

It follows that

$$
\boxed{\text{the vector } \vec{a} = \sum_{k=1}^{n} \alpha_k \vec{e}_k^{\,*} \text{ is perpendicular to the isograms of } \underline{a} = \sum_{k=1}^{n} \alpha_k \omega^k.}
$$

To summarize:

It is easy to exhibit the vector $\vec{a}$ corresponding to a given linear function $\underline{a} = \alpha_k \omega^k$, even *relative to an oblique coordinate basis*:
From the given basis $B = \{\vec{e}_j\}_{j=1}^n$ compute (using Eq.(29)) the elements of the reciprocal basis $R^* = \{\vec{e}_j^{\,*}\}_{j=1}^n$. The sought-after vector is simply

$$\vec{x} = \sum_{k=1}^{n} \alpha_k \vec{e}_k^{\,*}.$$

This vector is perpendicular to *all* the isograms of the given linear function $\underline{a}$.

8.2 Mathematization of the Law of X-ray Diffraction by a Crystal

A beam of X-rays striking the periodic lattice of a crystal gets refracted into discrete directions according to Bragg's Law. For a three-dimensional crystal this law is mathematized as follows.

The crystal consists of atoms arranged periodically into a lattice. They are located in a 3-d vector space. Its basis vectors $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ are the displacements into three different directions from the origin, the location of an arbitrarily chosen reference atom in the crystal. These displacements are determined by the three neighboring atoms closest to the atom at the origin.

All atoms of the crystal are located at some integral multiple linear combination of the basis vectors. Thus a typical atom is located at

$$\vec{x} = h\vec{e}_1 + k\vec{e}_2 + \ell\vec{e}_3 \qquad (h, k, \ell \in \{\text{integers}\})$$

As is the case for an isoclinic crystal, the basis

$$B = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \tag{36}$$

is oblique in general: the basis vectors need not be orthogonal nor of unit length.

Consider a beam of electromagnetic radiation (X-rays). For a plane wave this disturbance is characterized by its amplitude profile

$$\psi(\vec{x}) = Ae^{i\underline{\phi}(\vec{x})}; \quad (\nabla^2 + k^2)\psi = 0.$$

Here

$$\underline{\phi}(\vec{x}) = k_1 x^1 + k_2 x^2 + k_3 x^3.$$

is the value of the phase $\underset{\sim}{\phi}$ at location

$$\vec{x} = x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3.$$

It follows that relative to the dual basis

$$B^* = \{\omega^1, \omega^2, \omega^3 : \langle\omega^j|\vec{e}_i\rangle = \delta_i^J\},$$

the plane wave phase function is

$$\underset{\sim}{\phi} = k_1\omega^1 + k_2\omega^2 + k_3\omega^3.$$

When such a plane wave enters a crystal, it is observed[3] that emerging from this crystal there are discrete plane wave beams. Their directions relative to the incident beam is determined entirely by the atomic crystal basis, Eq.(36), more precisely, by the set of parallel crystal planes.

Before expressing this deterministic relation in mathematical terms, one must first mathematize these crystal planes.

8.2.1 Mathematized Crystal Planes

This is done by first observing that each one is one of the isograms of

$$\underset{\sim}{f} = h\omega^1 + k\omega^2 + \ell\omega^3 \qquad (h, k, \ell \in \{\text{integers}\}) \tag{37}$$

The integers (h, k, l), which are understood to be relative prime (i.e. have no common integral divisor) are the *Miller indices* of a given set of parallel crystal planes. Such a linear combination of dual basis elements with relative prime integral coefficients we shall call a *Miller covector*. It is an element of the dual space and each one of its isograms passing through an integral linear combination of basis vectors has an integral value.

There is a on-to-one correspondence between a set of parallel crystal planes and the Miller covector corresponding to this set. The following problem illustrates this fact.

Problem
Find the Miller covector for the set of parallel crystal planes one of which contains the set of linearly independent vectors $S = \{\vec{e}_1 + \vec{e}_2, 2\vec{e}_2, 3\vec{e}_3\}$.
Solution
This is a two-step process.

[3]Observed and explained by father and son W.H. Bragg and W.L. Bragg in 1913.

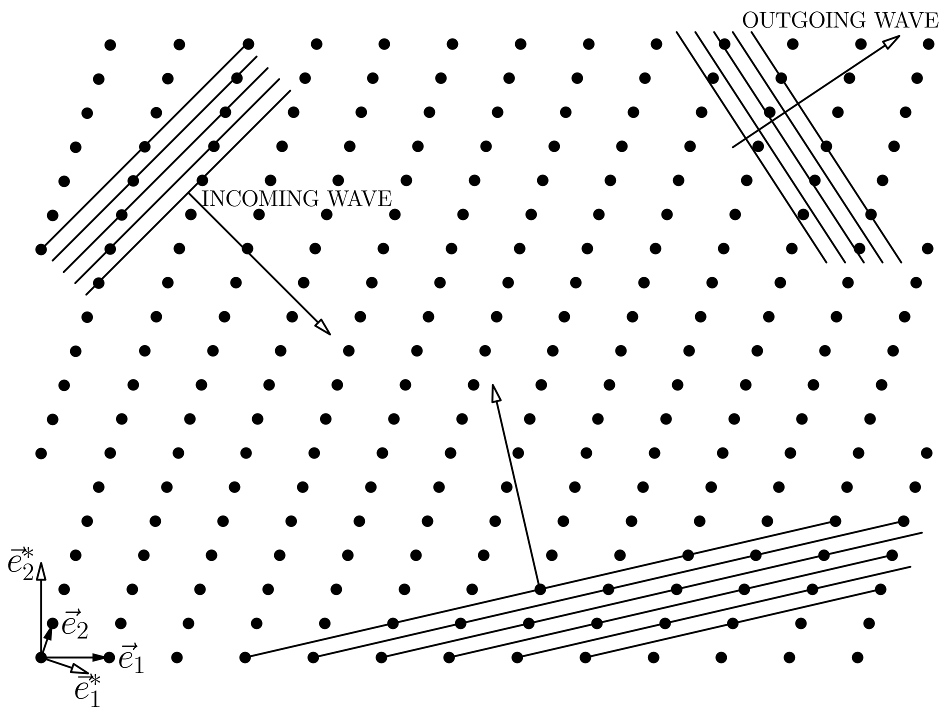

Figure 8: Crystal lattice with incoming and outgoing phase fronts. The outgoing wave is the result of a diffraction process. The difference between the in- and outgoing wave front normals is the vector perpendicular to the crystal planes indicated near the bottom. The basis vectors $\vec{e}_1$ and $\vec{e}_2$ are generated by directed pairs of nearest neighbor atoms (black circles).

Step 1: Let

$$\underline{g} = a\omega^1 + b\omega^2 + c\omega^3,$$

and find a, b, c so that $\underline{g}$ has an isogram , say $\underline{g} = 1$, passing through $\vec{e}_1 + \vec{e}_2, 2\vec{e}_2$, and $3\vec{e}_3\}$:

$$\begin{aligned} \underline{g}(\vec{e}_1 + \vec{e}_2) &= a + 3b + 0 = 1 \\ \underline{g}(2\vec{e}_2) &= 0 + 2b + 0 = 1 \\ \underline{g}(3\vec{e}_2) &= 0 + 0 + 3c = 1 \end{aligned}$$

Thus

$$\underline{g} = -\frac{1}{2}\omega^1 + \frac{1}{2}\omega^2 + \frac{1}{3}\omega^3$$

Step 2: The Miller indices are mutually prime integers. Thus multiplying $\underline{g}$ by the least common denominator yield the Miller covector,

$$\underline{f} = -3\omega^1 + 3\omega^2 + 2\omega^3 \tag{38}$$

The Miller indices of $\underline{f}$ are therefore

$$(h, k, \ell) = (-3, 3, 2).$$

8.2.2 Mathematized Diffraction Law

Focus on two X-ray beams and their respective phase functions $\underline{\phi}$,

$$\begin{aligned} \psi(\vec{x})^{incident}: \quad & \underline{\phi}^{inc} \quad = k_1^{inc}\ \omega^1 + k_2^{inc}\ \ \omega^2 + k_3^{inc}\ \ \omega^3 \\ \psi(\vec{x})^{diffracted}: \quad & \underline{\phi}^{diff} = k_1^{diff}\omega^1 + k_2^{diff}\omega^2 + k_3^{diff}\omega^3. \end{aligned} \tag{39}$$

To be diffracted by the set of crystal planes whose Miller covector is Eq.(37), the phase function of the diffracted beam must satisfy

$$\begin{aligned} \underline{\phi}^{inc} - \underline{\phi}^{diff} &= \underline{f} \\ \Delta k_1\omega^1 + \Delta k_2\omega^2 + \Delta k_3\omega^3 &= h\omega^1 + k\omega^2 + \ell\omega^3 \end{aligned} \tag{40}$$

where

$$\Delta k_1 \equiv (k_1^{inc} - k_1^{diff} = h \tag{41}$$

$$\Delta k_2 \equiv (k_2^{inc} - k_2^{diff}) = k \tag{42}$$

$$\Delta k_3 \equiv (k_3^{inc} - k_3^{diff}) = \ell \tag{43}$$

are the components of the difference between the incident and the diffracted phase functions (in physics, a.k.a. "propagation covectors") $\underline{\phi}^{inc}$ and $\underline{\phi}^{diff}$. They also equal the $\{\vec{e}_1^*, \vec{e}_2^*, \vec{e}_3^*\}$-basis components of the normals to the set of crystal planes, Eq.(40). The atoms in these planes are responsible for the particular phase fronts, Eq.(39). The three conditions, Eq.(41)-(43), for Bragg diffraction are known as Laue's equations.

8.3 Sampling Theorem as a Corollary to the Duality Principle

All observations and measurements processed by our mind into concepts and knowledge are finite. Concepts such as "infinity", "limit", "continuity", "derivative", etc., are not metaphysical[4] attributes of the world, but instead are mathematical methods. They are objective in that their composite nature reflects their nature of the world and the nature of our mind in grasping it.

One of the most ubiquitous concepts in the hierarchical network of mathematical methods is that of functions continuous on, say, the interval $[0, 2\pi]$. They form an infinite-dimensional vector space, which subsumes an unlimited number of different kinds of finite-dimensional vector spaces. Among them are those subspaces which are spanned by bases that reflect the particular manner of observation or measurement, specifically those those in which a function is sampled at equal intervals, say,

$$x_k = \frac{2\pi}{2N+1} k \qquad k = 0, 1, \cdots, 2N.$$

Recall that a chosen basis for a given vector space induces a unique set of linear functions. They are the coordinate functions on this vector space. These functions are also vectors. In fact, they form a basis, but for the dual vector space, which is entirely distinct. Its dimension is the same as that of the given vector space. The duality relation, Eq.(14) on page 12 mathematizes the duality principle. The sampling theorem is an application of the dual space concept.

Example 1 (Sampling a Band-Limited Function)
GIVEN:

a) The vector space of band-limited functions of period 2π,

$$V = \{f : \; f(x) = \sum_{m=0}^{N} a_m \cos mx + \sum_{m=1}^{N} \sin mx\} \equiv \mathcal{B}_N.$$

This is a $(2N+1)$-dimensional space with its standard trigonometric basis

$$B_{trig} = \{1, \cos mx, \sin mx : \; m = 1, \cdots, N\}, \tag{44}$$

[4] in the Greek sense, pertaining to the nature of reality.

or its exponential basis

$$B_{exp} = \{e^{imx}\}_{m=-N}^{N}. \tag{45}$$

b) The values $f(x_k)$ of f at the sampling points $\{x_k = \frac{2\pi}{2N+1}k$ with $k = 0, 1, \cdots, 2N\}$

FIND:

The function $f(x)$ for *all* x in terms of its known sampled values $\{f(x_k)\}_{k=0}^{2N}$.

SOLUTION:
The task at hand consists of answering the following question: Can one reconstruct f over the whole x-domain from one's knowledge of the f-values at the $2N + 1$ sample points x_k only? If, yes, HOW?
COMMENT:

This question cannot be answered without specifying a particular $(2N+1)$-dimensional subspace of $C[0, 2\pi]$, the infinite-dimensional subspace of functions continuous on $[0, 2\pi]$.

There are many such subspaces, and $V = \mathcal{B}_N$, the above space of band limited 2π-periodic functions of the present Example 1, is only one of them. Another one, considered in Example 2, below, on page 39, is $V = CPL[0, 2\pi]$, the $(2N + 1)$-dimensional subspace of continuous functions piecewise linear on the closed interval $[0, 2\pi]$.

In both subspaces a vector is specified by the same $2N + 1$ values of the sampled function. However, inbetween its sampling points, the function is interpolated in entirely different ways. The two subspaces are entirely different, but their dimensions are the same.

The answer to the posed question is that for sampling purposes the bases (44) or (45) on page 34 do not give good representations of elements in V. Instead, we construct x_k-localized functions by means of the following linear superpositions

$$\vec{e}_k(x) = \frac{1}{2N+1}\sum_{m=-N}^{N} e^{im(x-x_k)},\ \text{where } x_k = \frac{2\pi}{2N+1}k \text{ with } k = 0, 1, \cdots, 2N; \tag{46}$$

$$= 1 + \frac{1}{2N+1}\sum_{m=1}^{N} \cos m(x - x_k) \tag{47}$$

These are also band-limited functions, vectors in V. In fact, being mere geometrical series, their summed values are

$$\vec{e}_k(x) = \frac{1}{2N+1}\frac{\sin\left(N+\frac{1}{2}\right)(x-x_k)}{\sin\left(\dfrac{x-x_k}{2}\right)}. \tag{48}$$

These functions are x_ℓ-localized. They satisfy

$$\vec{e}_k(x_\ell) = \delta_{k\ell} \equiv \begin{cases} 0 & \ell \neq k \\ 1 & \ell = k \end{cases}.$$

Their graphs are exhibited in Figure 9 and they form a basis for V,

$$C = \{\vec{e}_0, \vec{e}_1, \cdots, \vec{e}_{2N}\}.$$

The reason for introducing this basis is that (i) sampling a function at a particular point

$$x_\ell = \frac{2\pi}{2N+1}\,\ell \quad \ell = 0, 1, \cdots, 2N+1$$

constitutes a linear map on the space of functions $f \in V = \mathcal{B}_N$:

$$\omega^\ell(f) = f(x_\ell),$$

and that (ii) these linear maps, which comprise the set

$$\{\omega^\ell\}_{\ell=0}^{2N},$$

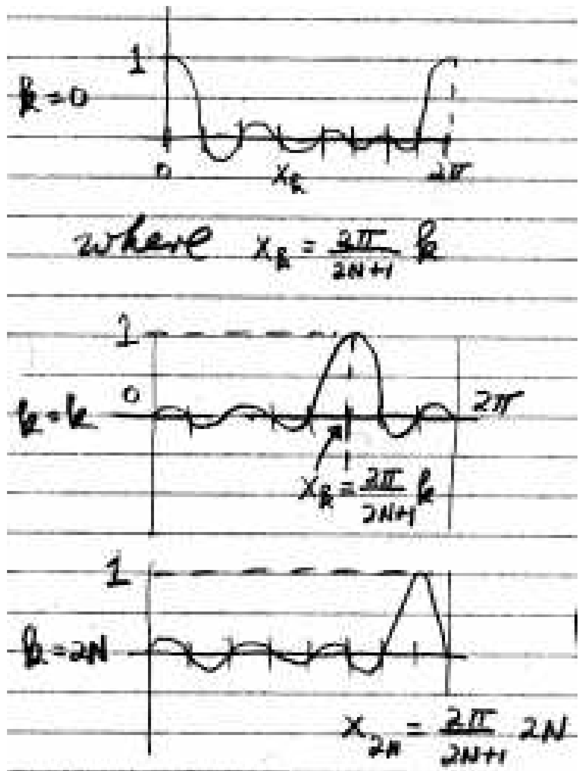

Figure 9: Graphs of x_k-localized functions $\vec{e}_k(x)$. The set $\{\vec{e}_k\}_{k=0}^{2N} \equiv B$ forms an alternative linearly independent spanning set (basis) for V.

have the property that

$$\omega^\ell(e_k) = e_k(x_\ell) = \delta_{\ell k}.$$

This is the duality relation. Thus the set of sampling maps

$$\{\omega^0, \omega^1, \omega^2, \cdots, \omega^{2N}\}$$

are precisely the basis elements dual to the constructed x_k-localized basis

$$\{e_0, e_1, e_2, \cdots, e_{2N}\}$$

as given by Eq.(48).

The reason for introducing this particular basis comes from our goal to characterize an arbitrary band-limited $f \in V = \mathcal{B}_N$ in terms of its sampled values at $x = x_k,\ k = 0, 1, \cdots, 2N$:

$$f \stackrel{\mathcal{B}_N}{\rightsquigarrow} \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{2N}) \end{bmatrix}_{\mathcal{B}_N}$$

There are $2N+1$ sampled values for each and every $f \in V$. This circumstance is mathematized by means of $2N+1$ sample-valued maps $\omega^0, \omega^1, \cdots, \omega^{2N}$ on V, namely

$$\begin{aligned} \omega^0(f) &= f(x_0) \\ \omega^1(f) &= f(x_1) \\ &\vdots \\ \omega^\ell(f) &= f(x_\ell) \\ &\vdots \\ \omega^{2N}(f) &= f(x_{2N}). \end{aligned}$$

These linear maps are precisely the coordinate functionals $\{\omega^\ell\}_{\ell=0}^{2N}$ induced by the vector basis $C = \{\vec{e}_k\}_{k=0}^{2N}$. This claim is validated by the fact that, according to Eq.(48) on page 36, the sampled values of the $\vec{e}_k$'s are

$$\vec{e}_k(x_\ell) = \delta_{k\ell} \equiv \begin{cases} 0 & \ell \neq k \\ 1 & \ell = k \end{cases},$$

in other words,

$$\omega^\ell(\vec{e}_k) \equiv \langle \omega^\ell | \vec{e}_k \rangle = \delta_{\ell k}$$

This is the duality relation: the basis $C = \{\vec{e}_k\}_{k=0}^{2N}$ has as its dual the set of basis (linear) functionals $C^* = \{\omega^\ell\}_{\ell=0}^{2N} \subset V^*$.

1. This set has two distinguishing properties:

    (a) On one hand each ω^ℓ *samples* any $f(x) \in V$ at $x = x_\ell$, and thereby yields $\omega^\ell(f) = f(x_\ell)$, the ℓ^{th} coordinate of $f \in V$ relative to B;

(b) on the other hand, and at the same time, each ω^ℓ *is a covector* which, together with the other elements in B^*, forms that basis for V^* which is dual to B.

2. The success of the sampling theorem hinges on the existence of the above two features:

   (a) The $\vec{e}_k$'s must form a basis for V. Consequently, one has

$$f(x) = \sum_{k=0}^{2N} \alpha_k \vec{e}_k,$$

   and

   (b) each $\vec{e}_k$ is a function with zero values at all equally spaced points x_ℓ, except one, where its value does not vanish. Consequently,

$$\underbrace{\omega^\ell(f)}_{f(x_\ell)} = \sum_{k=0}^{2N} \alpha_k \underbrace{\omega^\ell(\vec{e}_k)}_{\delta_k^\ell} = \sum_{k=0}^{2N} \alpha_k \underbrace{\vec{e}_k(x_\ell)}_{\delta_k^\ell}$$

   or

$$f(x_\ell) = \alpha_\ell$$

Consequently, $f(x)$ is given by

$$f(x) = \sum_{k=0}^{2N} f(x_k)\, \vec{e}_k,$$

a mathematically 100% accurate reconstruction of $f(x)$ in terms of its sampled values. This is the sampling theorem for band-limited functions $\mathcal{B}_N$.
Example 2 (Piecewise Linear Function via a Sampling Sequence)
GIVEN:

1. The closed interval $[x_0, x_n]$ which is partitioned by $x_0 < x_1 < \cdots < x_n$ into n *equally spaced* subintervals.

2. The values $y_0, y_1, \cdots, y_n$ of a function $f \in C[x_0, x_n]$ sampled at the above equally spaced points:

$$
\begin{aligned}
y_0 = &\quad f(x_0) \\
y_0 = &\quad f(x_1) \\
\vdots & \\
y_k = &\quad f(x_k) \\
\vdots & \\
y_n = &\quad f(x_n)
\end{aligned}
\tag{49}
$$

3. The set

$$
\begin{aligned}
CPL(\{x_0, x_1, \cdots, x_n\}) = \{\psi : \ &\psi \in C[x_0, x_n] \text{ and} \\
&\psi \text{ is linear on each subinterval } [x_{k-1}, x_k]\},
\end{aligned}
$$

which, being closed under addition and multiplication by scalars, is a subspace of $C[x_0, x_n]$.

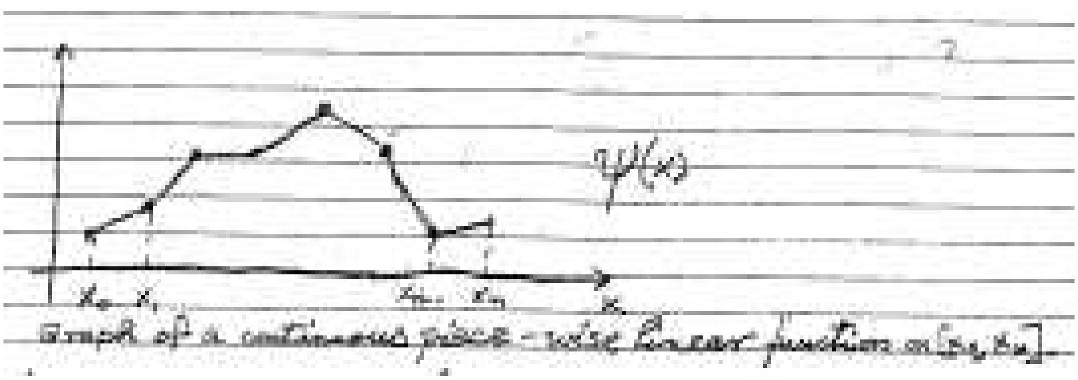

Figure 10: Graph of a CPL-function, an element of the vector space $CPL \subseteq C[x_0, x_n]$

EXHIBIT:

1. A basis for CPL whose elements $\psi_k(x)$ (like those of Eq.(48)) are x_ℓ-localized:

$$
\psi_k(x_\ell) = \delta_{k\ell} \equiv \begin{cases} 0 & \ell \neq k \\ 1 & \ell = k \end{cases}.
$$

2. The dual basis $\{\omega^j\}_{j=1}^n$ for CPL^*

3. For the given sampling sequence, Eq.(49) of the function $f \in C[x_0, x_n]$, the function $\psi(x) \in CPL$ such that

$$\psi(x_0) = y_0$$
$$\vdots$$
$$\psi(x_n) = y_n$$

SOLUTION:
1-2:

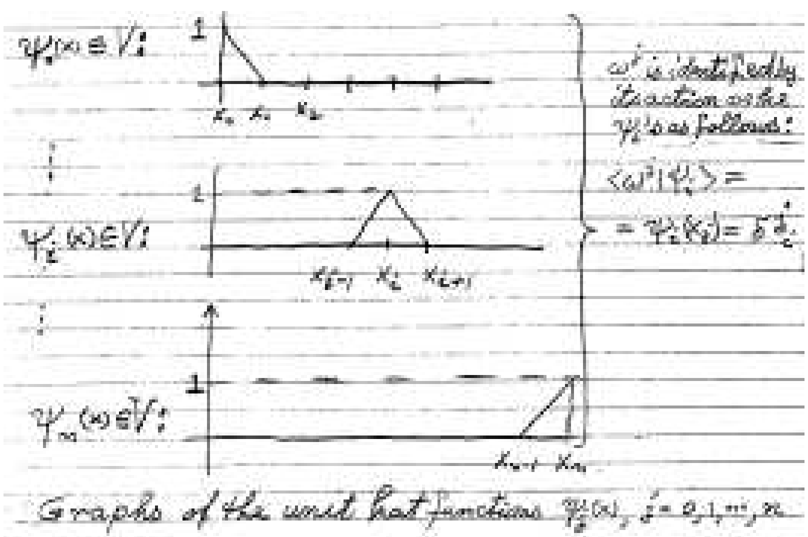

Figure 11: Graphs of the x_k-localized unit roof functions. They comprise a vector basis for CPL.

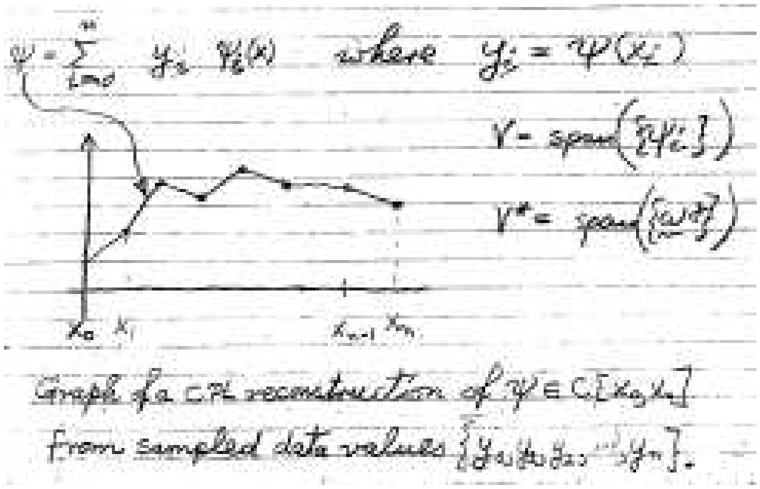

Figure 12: Graph of CPL-interpolation of sampled data $y_0, y_1, \cdots, y_n$. This interpolation has "two feet": as a continous function it has one foot in analysis, as a function which is linear it has the other foot in linear algera of dealing with CPL vector spaces.

9 PROBLEMS

1. (DUAL BASIS AS A SET OF MULTIVARIABLE FUNCTIONS)
   Let $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a basis for R^3 defined by

$$\vec{v}_1 = (1,0,1)^t$$
$$\vec{v}_2 = (1,1,1)^t$$
$$\vec{v}_3 = (2,2,0)^t$$

   a) FIND the basis $\{f, g, h\}$ of linear functions (i.e. row vectors) dual to B.
   b) EXHIBIT

$$f(\vec{x}) = f(x,y,z)$$
$$g(\vec{x}) = g(x,y,z)$$
$$h(\vec{x}) = h(x,y,z)$$

2. (VECTOR BASES AND THEIR DUAL BASES)

Consider 3-dimensional vector space spanned by

$$\vec{e}_1 = \vec{i} + \vec{j} + \vec{k}$$
$$\vec{e}_2 = -\vec{i} + \vec{j} + \vec{k}$$
$$\vec{e}_3 = -\vec{i} - \vec{j} + \vec{k}$$

where $\vec{i}, \vec{j}, \vec{k}$ are the usual orthogonal basis vectors.

(a) If $\{\underline{\tau}^1, \underline{\tau}^2, \underline{\tau}^3\}$ is the basis *dual* to $\{i, j, k\}$, i.e.

$$\begin{array}{lll} \langle \underline{\tau}^1 | \vec{i} \rangle = 1 & \langle \underline{\tau}^1 | \vec{j} \rangle = 0 & \langle \underline{\tau}^1 | \vec{k} \rangle = 0 \\ \langle \underline{\tau}^2 | \vec{i} \rangle = 0 & \langle \underline{\tau}^2 | \vec{j} \rangle = 1 & \langle \underline{\tau}^2 | \vec{k} \rangle = 0 \\ \langle \underline{\tau}^3 | \vec{i} \rangle = 0 & \langle \underline{\tau}^3 | \vec{j} \rangle = 0 & \langle \underline{\tau}^3 | \vec{k} \rangle = 1 \end{array}$$

FIND the basis $\{\underline{\omega}^1, \underline{\omega}^2, \underline{\omega}^3\}$ *dual* to $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

(b) Let h, k, ℓ be three scalars.
FIND that linear function, call it $\underline{f}$, which has the value 1 at each of the three points $\dfrac{\vec{e}_1}{h}, \dfrac{\vec{e}_2}{k}, \dfrac{\vec{e}_3}{\ell}$.
Thus *write down* this function in terms of $\{\underline{\tau}^1, \underline{\tau}^2, \underline{\tau}^3\}$ and in terms of $\{\underline{\omega}^1, \underline{\omega}^2, \underline{\omega}^3\}$.

(c) FIND the set of reciprocal basis vectors (e_1^*, e_2^*, e_3^*), which satisfy

$$\vec{e}_i \cdot \vec{e}_j^{\,*} = \delta_{ij} \, .$$

(Here "·" is the familiar inner product obtained from $\vec{i} \cdot \vec{i} = 1$, $\vec{i} \cdot \vec{j} = 0$, $\vec{i} \cdot \vec{k} = 0$, etc.)

(d) What relation, if any, does there exist between these basis vectors $\vec{e}_1^{\,*}$, $\vec{e}_2^{\,*}$, and $\vec{e}_3^{\,*}$ and the level surfaces of $\underline{\omega}^1$, $\underline{\omega}^2$, and $\underline{\omega}^3$?

(e) FIND the unit vector perpendicular to the level surface $\underline{f} = 1$.

(f) Write down the distance from the origin to $\underline{f} = 1$.
REMINDER: If you get bogged down in detailed computation, you are not making optimal use of the nature of the dual basis!

<u>Comment:</u> The components of $\underline{f}$ found in (b) relative to $\{\underline{\omega}^i\}$ are the *Miller indices* of a set of parallel planes in a crystal whose primitive translation vectors are $\vec{e}_1, \vec{e}_2, \vec{e}_3$.
SEE C. KITTEL, Introduction to SOLID STATE PHYSICS.

3. (DEFINITE INTEGRALS DUALS ON THE SPACE OF POLYNOMIALS) Let $V = \mathcal{P}_2$ be the vector space of all polynomial functions p from R into R which have degree 2 or less:

$$p(x) = a_0 + a_1 x + a_2 x^2 \,.$$

Define the following linear functionals on V by

$$\begin{aligned} \langle f_1 \mid p \rangle \equiv f_1(p) &= \int_0^1 p(x)dx \\ \langle f_2 \mid p \rangle \equiv f_2(p) &= \int_0^2 p(x)dx \\ \langle f_3 \mid p \rangle \equiv f_3(p) &= \int_0^{-1} p(x)dx \,. \end{aligned}$$

SHOW that $\{f_1, f_2, f_3\}$ is a basis for V^* by exhibiting the basis for V of which it is the dual.