

LECTURE 10

10.1

Acceleration as Fermi-Walker transport

- I. Moving vs. fixed frames
- II. Fermi-Walker transport
- III. Fermi-Walker transport of a vector

TW [Ch. 6]

I. Moving Frame vs. Fixed Frame

10.2

The mathematization of accelerative change in the motion of particles/bodies gives rise to parametrized families of inertial reference frames, each one with an observer-defined tetrad of 4-vectors. They form a basis for the 4-d vector space whose origin is on the observer's world line. For an observer subject to uniform linear acceleration this world line is

$$X(\tau): \{x^0(\tau), x^1(\tau), x^2(\tau), x^3(\tau)\} = \left\{ \frac{1}{g} \operatorname{sh} g\tau, \frac{1}{g} \operatorname{ch} g\tau, 0, 0 \right\},$$

and the corresponding parametrized family of Lorentz orthonormal frames is

$$u: (u)^\mu \equiv (e_0)^\mu = (\operatorname{ch} g\tau, \operatorname{sh} g\tau, 0, 0)$$

$$\frac{a}{g}: \left(\frac{a}{g}\right)^\mu \equiv (e_1)^\mu = (\operatorname{sh} g\tau, \operatorname{ch} g\tau, 0, 0)$$

$$e_{2'}: (e_{2'})^\mu = (0, 0, 1, 0)$$

$$e_{3'}: (e_{3'})^\mu = (0, 0, 0, 1)$$

$$\left(\begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$$

The components of this moving frame, $\{e_0(\tau), e_1(\tau), e_{2'}(\tau), e_{3'}(\tau)\}$, are typically given relative to a physically fixed basis

$$e_0: (1, 0, 0, 0)$$

$$e_1: (0, 1, 0, 0)$$

$$e_2: (0, 0, 1, 0)$$

$$e_3: (0, 0, 0, 1).$$

Its components are fixed because these basis vectors are aligned with the "fixed" stars.

II. Fermi-Walker Transport

The physically essential property of the moving frame, Eq.(10.1), associated with a linear uniformly accelerated observer is that it is non-rotating. Three freely gimballed gyroscopes aligned along the three spatial coordinate axes $\bar{x}, \bar{y}, \bar{z}$ in the rocket frame would not precess in the frame of the accelerating rocket as it executed its spacetime trajectory.

Fermi-Walker transport extends this non-rotation feature to a rocket subject to arbitrary acceleration.

1.) Fermi-Walker transport: Its construction.

A. GIVEN:

(i) The spacetime trajectory

$$X(\tau): \{x^0(\tau), x^1(\tau), x^2(\tau), x^3(\tau)\}$$

of a particle/body subjected to some given acceleration.

(ii) The globally defined inertial LAB frame whose spatial axes e_1, e_2, e_3 point into the direction of some three chosen fixed stars

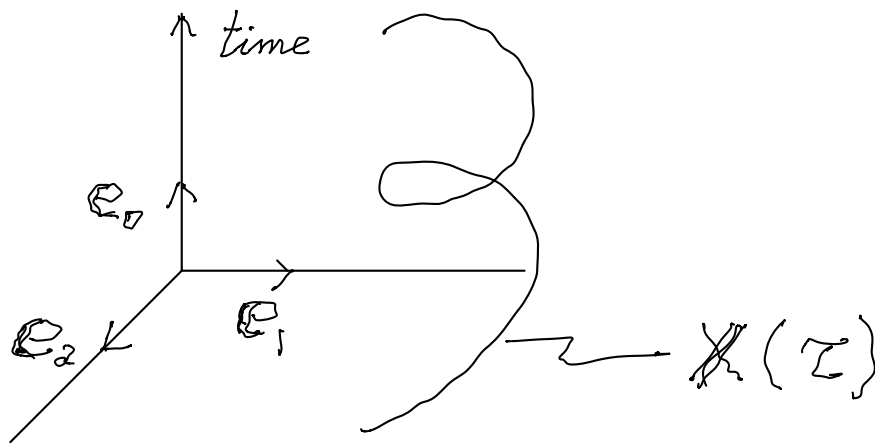


Figure 10.1: Inertial LAB frame with fixed orientation relative to the FIXED STARS.

(i) Three GYROSCOPES attached to and carried along $X(\tau)$, the accelerated world line, but gimballed freely so that no torques act on each of the gyroscopes.

(ii) The τ -parametrized gyroscope-induced Lorentz orthonormal family of frames

$$\{e_{0'}(\tau) = \frac{dX(\tau)}{d\tau} \equiv u(\tau), e_{1'}(\tau), e_{2'}(\tau), e_{3'}(\tau)\}$$

which consist of

a) the particle's 4-velocity

$$u(\tau) = \frac{dX}{d\tau} : \left\{ u^\mu(\tau) = \frac{dx^\mu}{d\tau}, \mu = 0, 1, 2, 3 \right\}$$

and

b) the three orthonormal directions determined by the three respective gyroscope directions.

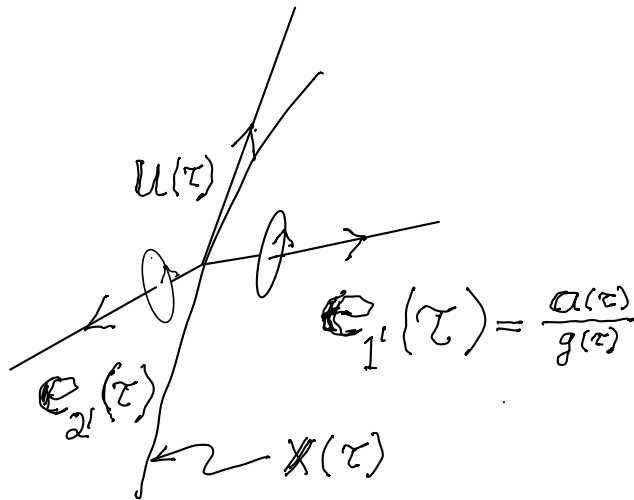


Figure 10.2: Gyroscope and 4-velocity determined Lorentz frame with one of the gyroscope axes aligned along the direction of the local acceleration vector a

(v) Notation:

Relative to the global LAB frame the gyroscope-determined 4-vectors have the components

$$e_{1'} : \{e_{1'}^{\mu} : \mu = 0, 1, 2, 3\}$$

$$e_{2'} : \{e_{2'}^{\mu} : \mu = 0, 1, 2, 3\}$$

$$e_{3'} : \{e_{3'}^{\mu} : \mu = 0, 1, 2, 3\}$$

$$u = e_{0'} : \{u^{\mu} : \mu = 0, 1, 2, 3\}$$

vs.

relative to the comoving frame

these 4-vectors have the components

$$e_{1'} : \{e_{1'}^{\mu'} : \mu = 0, 1, 2, 3\} = \{0, 1, 0, 0\}$$

$$e_{2'} : \{e_{2'}^{\mu'} : \mu = 0, 1, 2, 3\} = \{0, 0, 1, 0\}$$

$$e_{3'} : \{e_{3'}^{\mu'} : \mu = 0, 1, 2, 3\} = \{0, 0, 0, 1\}$$

$$u = e_{0'} : \{u^{\mu'} : \mu = 0, 1, 2, 3\} = \{1, 0, 0, 0\}$$

B. CONCLUSION:

10.6

(i) The gyroscopes form a spatial platform with an orthonormal basis:

$$e_{i'} \cdot e_{j'} \equiv e_{i'}^{\mu'} e_{j'}^{\nu'} \eta_{\mu'\nu'} = \eta_{i'j'} \quad \left(= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \begin{matrix} i' \\ j' \end{matrix} = 1, 2, 3 \right)$$

(ii) The time-like basis vector $e_{0'} \equiv u$ is Lorentz normalized,

$$e_{0'} \cdot e_{0'} = u \cdot u = u^{\mu'} u^{\nu'} \eta_{\mu'\nu'} = -1$$

and Lorentz orthogonal to the spatial platform:

$$e_{0'} \cdot e_{i'} \equiv u \cdot e_{i'} = 0 \quad i' = 1, 2, 3$$

(iii) Consequently,

$$e_{\mu'} \cdot e_{\nu'} = \eta_{\mu'\nu'} \quad \left(= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

is a set of 16 frame invariants, and

$$\{e_{0'}(\tau), e_{1'}(\tau), e_{2'}(\tau), e_{3'}(\tau)\} \equiv B_{FW}$$

is a τ -parametrized family of bases for the τ -parametrized family of instantaneous Lorentz frames (a.k.a. "moving frames").

(iv) Summary:

The τ -parametrized family B_{FW} is determined by the freely gimballed gyroscopes which are carried along a given accelerated worldline whose tangent is its instantaneous 4-velocity.

B_{FW} is called a Fermi-Walker basis

III. Fermi-Walker transport of a vector

Let $v(\tau)$ be a τ -parametrized family of vectors along a given curve $X(\tau)$ in spacetime

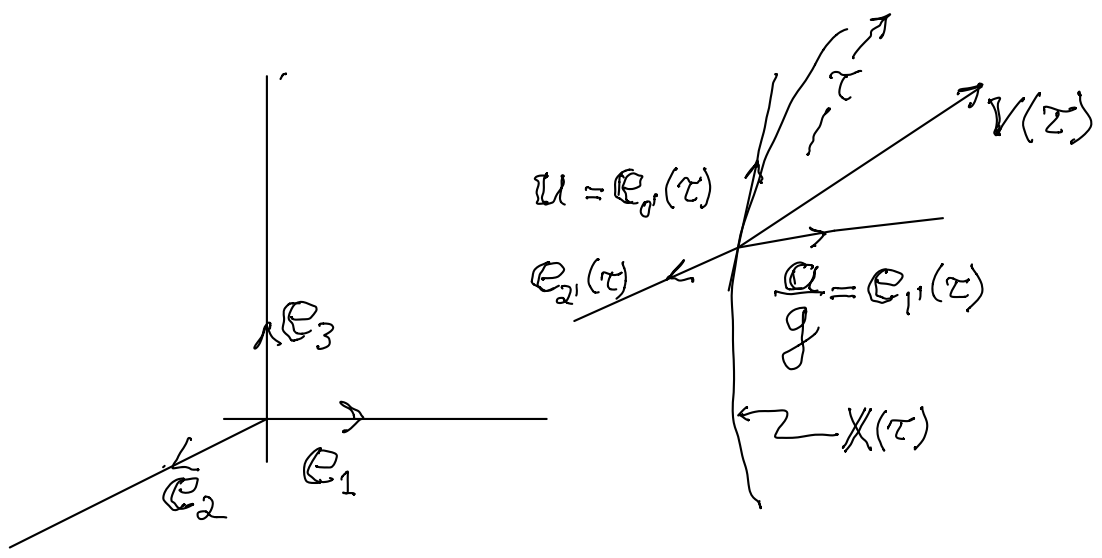


Figure 10.3: Instantaneous Fermi-Walker frame in a global Lab (= "fixed stars") frame.

Expand $v(\tau)$ in terms of the moving Fermi-Walker basis and in terms of the fixed Lab basis. These expansions lead to the following

Definition (F-W Transport)

A τ -parametrized family of vectors $v(\tau)$ is said to be Fermi-Walker transported along the given spacetime trajectory $X(\tau)$ if its components remain constant relative to the gyroscope-induced basis

$$v(\tau) = \underbrace{e_{\mu'}(\tau)}_{\text{constant}} v^{\mu'} = \underbrace{e_{\mu}}_{\text{fixed}} v^{\mu}(\tau) \quad \left(\begin{array}{l} \text{relative to} \\ \text{the fixed stars} \end{array} \right)$$

Thus, a Fermi-Walker transported family of vectors $V(\tau)$ have constant components relative to the F-W basis

(10.8)

$$\{e_{\mu'}(\tau)\}: V(\tau) = e_{\mu'}(\tau) v^{\mu'}$$

This gyroscope induced basis changes relative to the fixed stars. On the other hand, $V(\tau)$ has changing components relative to the fixed star basis

$$\{e_{\mu}(\tau)\}: V(\tau) = e_{\mu}(\tau) v^{\mu}(\tau)$$

III. The law governing Fermi-Walker transport

The mathematization of the gyroscope-induced F-W transport consists of setting up the system of differential equations whose solutions are the coefficients of $V(\tau)$ relative to the fixed star basis:

$$e_{\mu} v^{\mu}(\tau) = V(\tau) = e_{\mu'}(\tau) v^{\mu'}$$

?

MTW in their Section 6.5, p170-171, show that this system of differential equations is

$$\frac{dv^{\mu}}{d\tau} = (u^{\mu} a^{\nu} - a^{\mu} u^{\nu}) v_{\nu}, \quad \mu = 0, 1, 2, 3$$

or in terms of frame invariant coefficients

$$\frac{dv^{\mu}}{d\tau} = u^{\mu} a \cdot v - a^{\mu} u \cdot v.$$

Here

$$u^{\mu}(\tau) = \frac{dx^{\mu}}{d\tau} \quad (\text{"4-velocity"})$$

$$a^{\nu}(\tau) = \frac{d^2 x^{\nu}}{d\tau^2} \quad (\text{"4-acceleration"})$$

$$v_{\nu}(\tau) = \eta_{\nu\sigma} v^{\sigma}(\tau)$$

10.9

In MTW Ex. 6.9 is the problem of solving the F-W system for the case where $v(\tau) = S(\tau)$ is the spin 4-vector of a spinning electron in a circular orbit.