LECTURE 11

I. Fermi-Walker coordinates due to an accelerated frame

II. Coordinates of a uniformly accelerated frame.

III. Polar coordinates vs. pseudo-polar coordinates.

[MTW Sect. 6.6]

I. Curvilinear Coordinates Induced by a Uniformly Accelerated Frame.

The basis vectors F-W transported along a given world line give rise to a natural local curvilinear coordinate system. The coordinate lines are required to be tangent to the four respective F-W basis vectors emanating from the point event moving along the world line. Their construction is condensed into the following

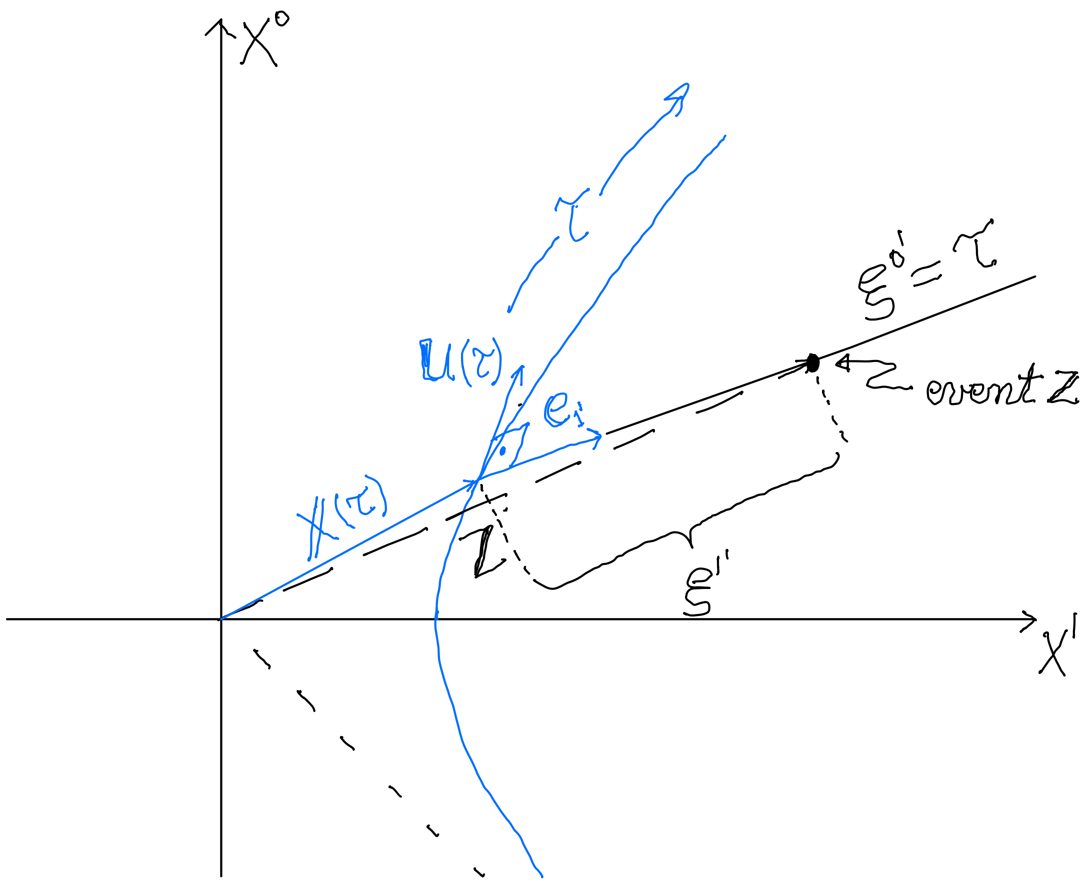

Figure 10.1: An event

$$\mathbb{Z} = \mathbb{e}_0 \xi^0 + \mathbb{e}_1 \xi^1 + \mathbb{e}_2 \xi^2 + \mathbb{e}_3 \xi^3$$

in the neighborhood of the spacetime trajectory

$$\mathbb{X}(\tau) = \mathbb{e}_0 X^0(\tau) + \mathbb{e}_1 X^1(\tau) + \mathbb{e}_2 X^2(\tau) + \mathbb{e}_3 X^3(\tau)$$

of an accelerated observer $\mathcal{O}'$ has Fermi-Walker coordinates
$(\xi^{0'}, \xi^{1'}, \xi^{2'}, \xi^{3'})$.

They are related to Z's rectilinear coordinates $\{z^0, z^1, z^2, z^3\}$ by the condition

$$Z = e_{1'}\xi^{1'} + e_{2'}\xi^{2'} + e_{3'}\xi^{3'} + X(\tau).$$

Definition ("Fermi-Walker coordinates")

(i) Let $X(\tau)$ be the world line of the given observer $\mathcal{O}$.

(ii) Let

$$\{e_{0'}(\tau), e_{1'}(\tau), e_{2'}(\tau), e_{3'}(\tau)\} \equiv \{e_{\alpha'}(\tau) : \alpha' = 0, 1, 2, 3\}$$

be a F-W transported tetrad of basis vectors,

i.e.

$$\frac{dv^\mu}{d\tau} = (u^\mu a^\nu - a^\mu u^\nu)\, v_\nu \quad \text{for } v_\nu = \eta_{\nu\sigma}(e_{\alpha'})^\sigma, \ \alpha' = 0, 1, 2, 3.$$

Then, given an event Z, $(\xi^{1'}, \xi^{2'}, \xi^{3'})$ are its spatial coordinates relative to $\mathcal{O}$ whenever

$$\boxed{Z = \xi^{1'} e_{1'} + \xi^{2'} e_{2'} + \xi^{3'} e_{3'} + X(\tau)}, \tag{11.1}$$

and its time coordinate $\xi^{0'}$ is given by the requirement that

$$\boxed{\xi^{0'} = \tau.} \tag{11.2}$$

As recorded by an observer in an inertial frame, the coordinates of a typical event Z are

$$Z : \{z^0, z^1, z^2, z^3\}.$$

On the other hand, relative to an accelerated frame the coordinates of that event are

$$Z: \{\xi^{1'}, \xi^{2'}, \xi^{3'}, \tau\}$$

Thus we have a 1-1 coordinate transformation,

$$\{z^0, z^1, z^2, z^3\} \longleftrightarrow \{\xi^{1'}, \xi^{2'}, \xi^{3'}, \tau\}$$

II. Coordinate system for a uniformly accelerated frame.

1. The given spacetime trajectory of the spatial origin of the spacial origin of the accelerated observer is

$$X(\tau) = \underbrace{g^{-1}\,\mathrm{sh}\,g\tau}_{X^0(\tau)}\, e_0 + \underbrace{g^{-1}\mathrm{ch}\,g\tau}_{X^1(\tau)}\, e_1 + \underbrace{0}_{X^2(\tau)}\, e_2 + \underbrace{0}_{X^3(\tau)}\, e_3$$

2. There four solutions the F-W equation

$$\frac{dv^\mu}{d\tau} = (u^\mu a^\nu - a^\mu u^\nu)\, v_\nu ; \qquad v_\nu = \eta_{\nu\sigma} v^\sigma,$$

namely,

$$\begin{aligned}
\{v^\mu\} &= \{u^\mu\} = \{v_{0'}^{\ \mu}\} = \{\mathrm{ch}\,g\tau, \mathrm{sh}\,g\tau, 0, 0\} \equiv \{e_{0'}^{\ \mu}(\tau)\} \\
\{v^\mu\} &= \{\tfrac{a^\mu}{g}\} = \{v_{1'}^{\ \mu}\} = \{\mathrm{sh}\,g\tau, \mathrm{ch}\,g\tau, 0, 0\} \equiv \{e_{1'}^{\ \mu}(\tau)\} \\
\{v^\mu\} &= \{0,0,1,0\} = \{v_{2'}^{\ \mu}\} = \{0, 0, 1, 0\} \equiv \{e_{2'}^{\ \mu}(\tau)\} \\
\{v^\mu\} &= \{0,0,0,1\} = \{v_{3'}^{\ \mu}\} = \{0, 0, 0, 1\} \equiv \{e_{3'}^{\ \mu}(\tau)\}
\end{aligned}$$

3. The boxed equation (11.1) on page 11.3, when expressed relative to the Lab basis $\{e_0, e_1, e_2, e_3\}$ is

$$\begin{aligned}
z^0 e_0 + z^1 e_1 + z^2 e_2 + z^3 e_3 &= X(\tau) + \xi^{1'} e_{1'} + \xi^{2'} e_{2'} + \xi^{3'} e_{3'} \\
&= g^{-1}\mathrm{sh}\,g\tau\, e_0 + g^{-1}\mathrm{ch}\,g\tau\, e_1 + \xi^{1'}(\mathrm{sh}\,g\tau\, e_0 + \mathrm{ch}\,g\tau\, e_1) + \xi^{2'} e_2 + \xi^{3'} e_3
\end{aligned}$$

Equating coefficients of the basis vectors e_μ yields the coordinate transformation between the rectilinear coordinates for an inertial frame and the curvilinear coordinates for a linear uniformly accelerated frame

$$(z^0\; z^1\; z^2\; z^3) \longleftrightarrow (\tau, \xi^{1'}, \xi^{2'}, \xi^{3'}),$$

namely,

$$\boxed{\begin{array}{ll} e_0: & z^0 = (g^{-1}+\xi^{1'})\,\mathrm{sh}\, g\tau \;\equiv \xi\,\mathrm{sh}\, g\tau \\ e_1: & z^1 = (g^{-1}+\xi^{1'})\,\mathrm{ch}\, g\tau \;\equiv \xi\,\mathrm{ch}\, g\tau \\ e_2: & z^2 = \xi^{2'} \\ e_3: & z^3 = \xi^{3'} \end{array}} \quad \left.\right\} 0<\xi<\infty \qquad (11.3)$$

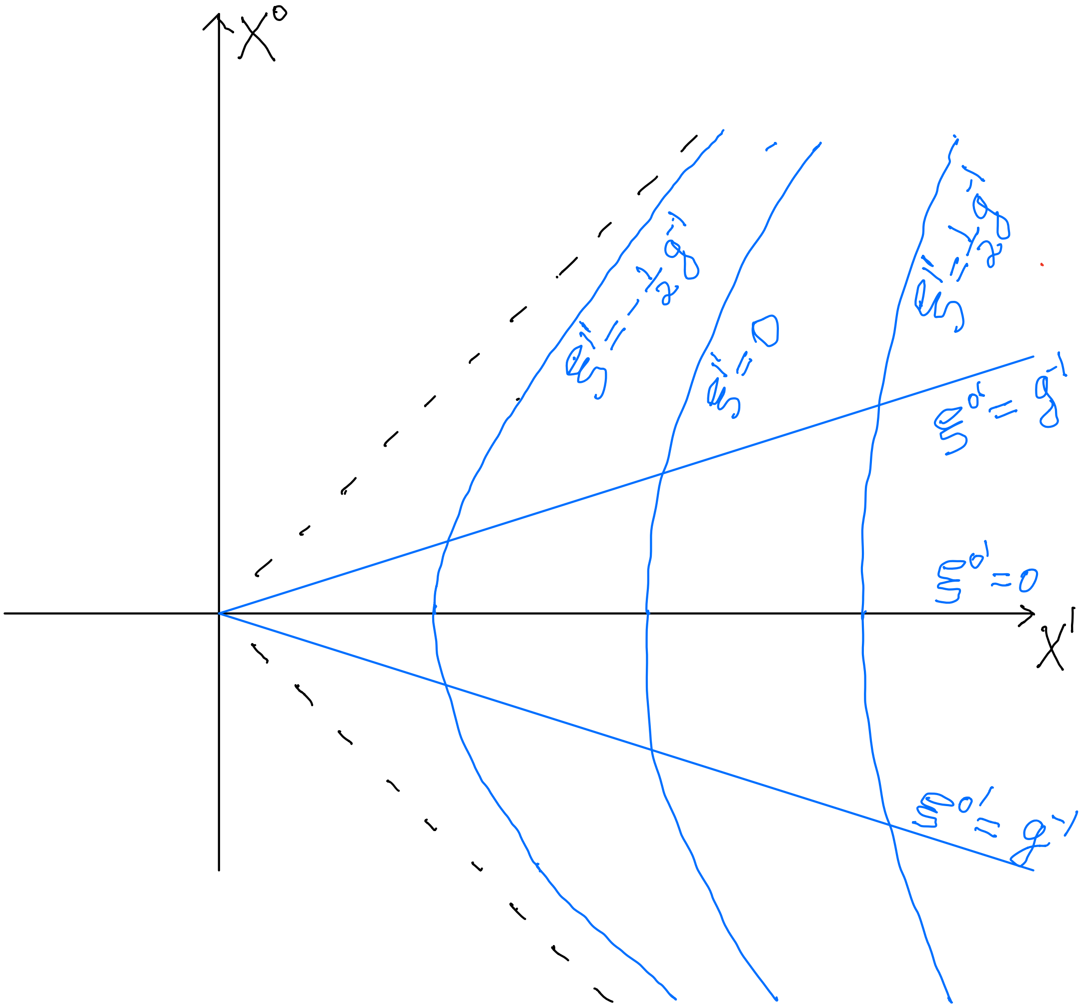

Figure 11.2: Fermi-Walker (="Rindler") coordinate system for a linear uniformly accelerated frame

$$x^0 = (\xi^{1'} + g^{-1})\,\mathrm{sh}\, g\tau \equiv \xi\, \mathrm{sh}\, g\tau$$
$$x = (\xi^{1'} + g^{-1})\,\mathrm{ch}\, g\tau \equiv \xi\, \mathrm{ch}\, g\tau$$

3. In light of coordinate transformation, Eq.(11.3) on page 11.5, the form of the squared invariant spacetime interval

$$(ds)^2 \equiv -(dz^0)^2 + (dz^1)^2 + (dz^2)^2 + (dz^3)^2 \qquad \left(\begin{array}{l}\text{rectilinear} \\ \text{a.k.a. "Minkowski"} \\ \text{coordinates}\end{array}\right)$$

assumes the form

$$\begin{aligned} ds^2 &= -(\xi^{1'} + g^{-1})^2 d\tau^2 + (d\xi^{1'})^2 + (d\xi^{2'})^2 + (d\xi^{3'})^2 \\ &= -\xi^2 d\tau^2 + (d\xi)^2 + dy^2 + dz^2 \end{aligned} \qquad \left(\begin{array}{l}\text{"Rindler"} \\ \text{coordinates}\end{array}\right)$$

Comment.

The construction of these coordinates is based on the locus of events, world lines, which are spacetime hyperbolas,

$$(x^1)^2 - (x^0)^2 = \xi^2.$$

These world lines of constant local acceleration correspond to what in Euclidean space are concentric circles

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} x^2 + y^2 = r^2$$

In Euclidean space these circles comprise the familiar polar coordinates, relative to which the invariant distance has the form

$$(d\sigma)^2 = (dx)^2 + (dy)^2 + dz^2$$
$$= r^2 d\theta^2 + dr^2 + dz^2$$

Thus the (ξ, τ) coordinates are sometimes called "pseudo polar" coordinates. However nowadays they are called Rindler coordinates, after Wolfgang Rindler who pointed out their utility and the fundamental role they play in spacetime physics.